Binary Search Trees

Instructors: Sam McCauley and Dan Barowy

April 20, 2022

Admin

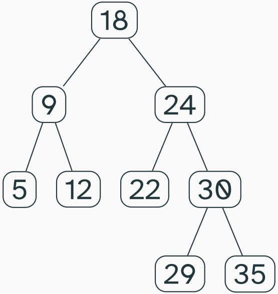
- Sign up to be a TA! Deadline Friday
  - End of the form asks to list professors; pretty much anyone you've had is probably fine
  - If you want we can have a brief conversation where I say I'm OK with you putting my name down
- Lab 8 tomorrow: please read over the lab and create a design document before your lab
  - We'll actually collect them this week
  - Very important to get a head start on the lab
- We'll briefly discuss course registration Friday
- Any questions?

Tree Iterators

Implementing Tree Iterators

- Goal: implement the traversals above as an iterator
- Can do `next()` and `hasNext()` on demand
- Problem: want to get values on demand (should be updated as the tree is updated)
  - Don't want to traverse the tree, store all tree values, and then dispense them one by one
  - Instead: each call to `next()` should go to the next node in the tree we want to output
- Challenge: implementing a recursive traversal piece-by-piece
- To think about: what data structure helps with recursion?

Pre-order traversal



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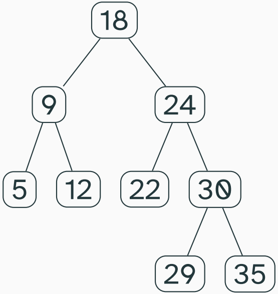

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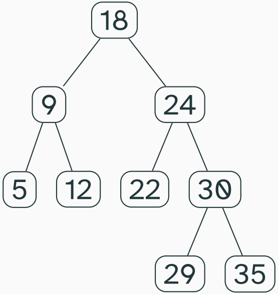

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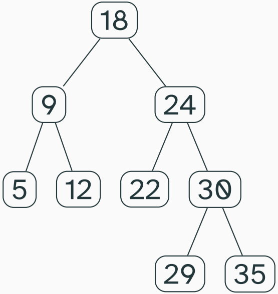

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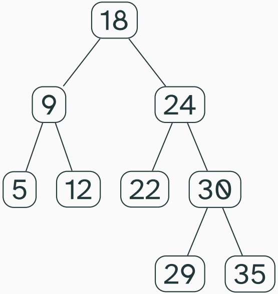

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- Instead: maintain nodes to visit on a stack!

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- `hasNext()`?
    - Just returns if the stack is empty

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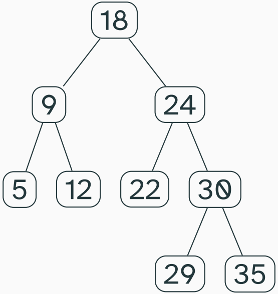

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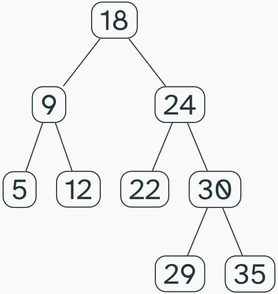

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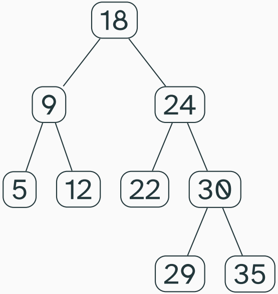

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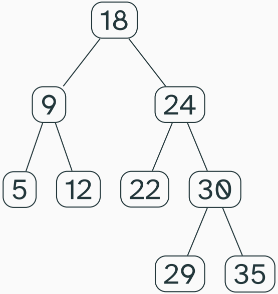

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- Nice idea, but it takes some care. Let's be a bit more specific

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- Let’s look at the code

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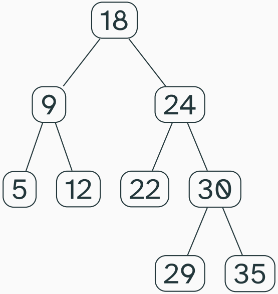

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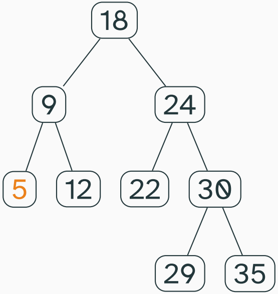

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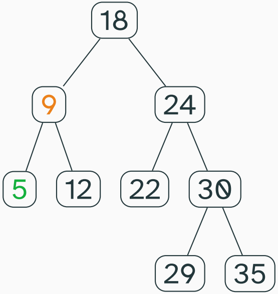

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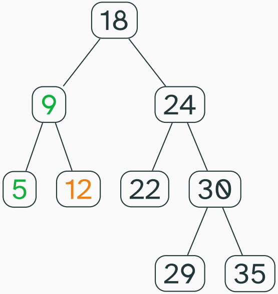

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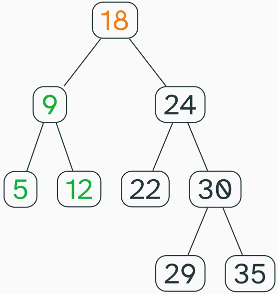

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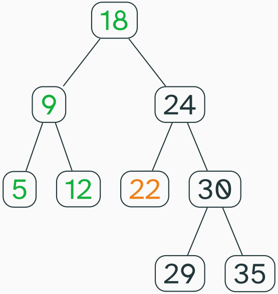

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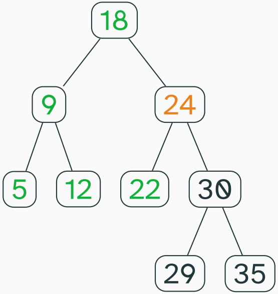

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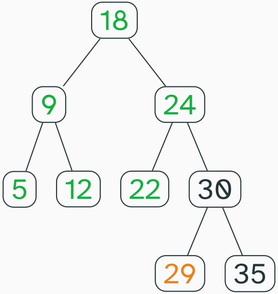

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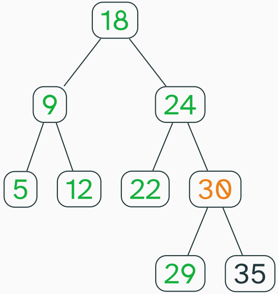

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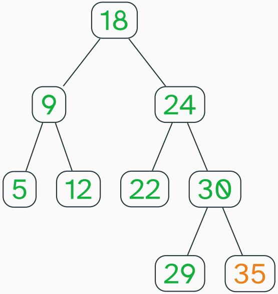

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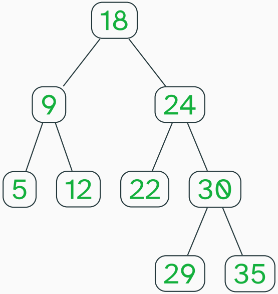

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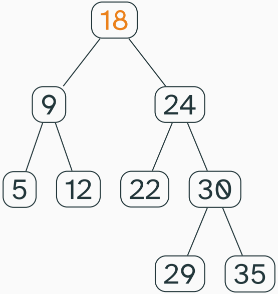
- If you pop a node, and it’s the left child of its parent, push the parent’s right child (and leftmost descendants) onto the stack

Level-order Traversal



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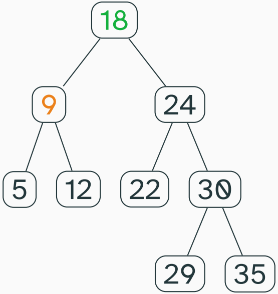

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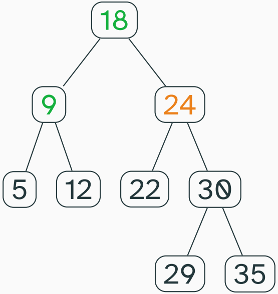

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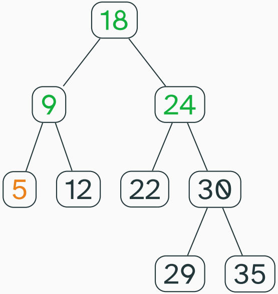

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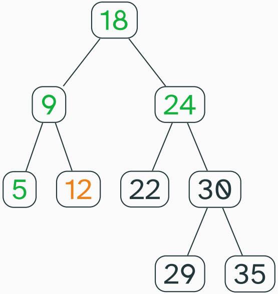

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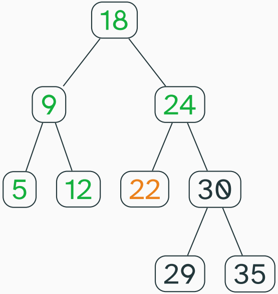

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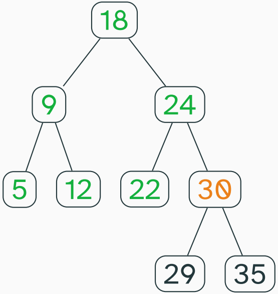

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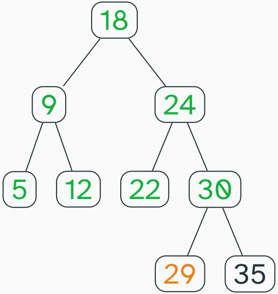

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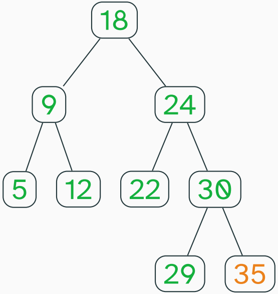

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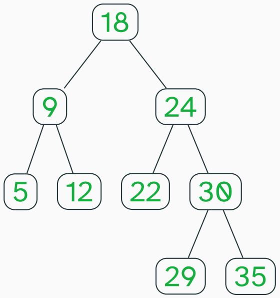

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- Key insight: the order we visit nodes at a given “level” is the same order we visited their parents
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- …Can we use a queue?

Level-order iterator

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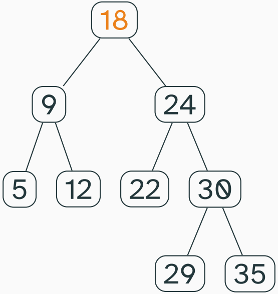

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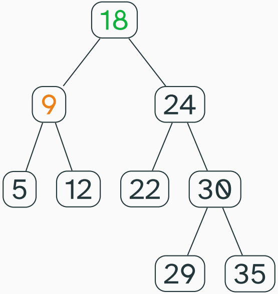

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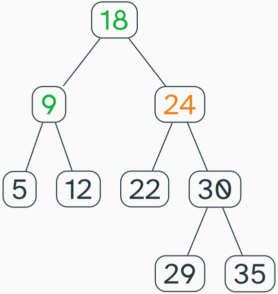

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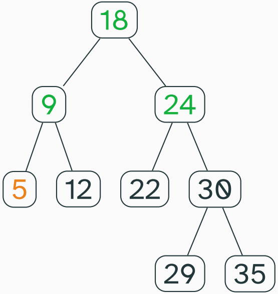

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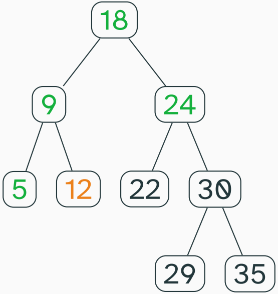

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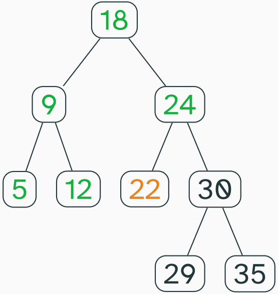

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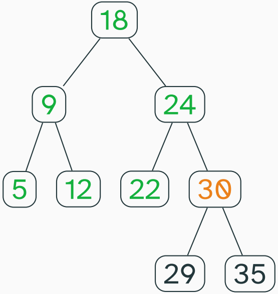

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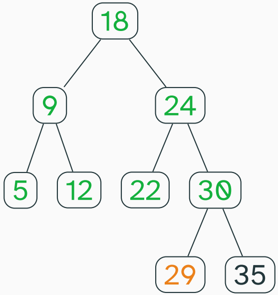

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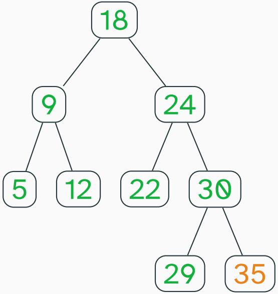

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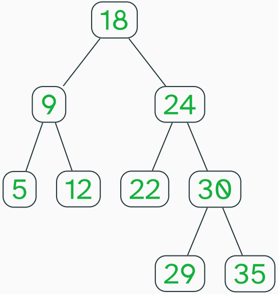

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- How can we do something similar for trees?

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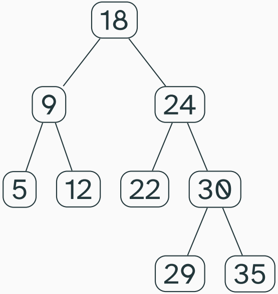

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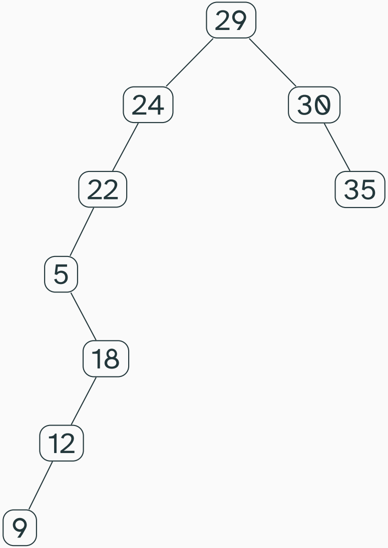

Is this a binary search tree? (It has the same elements!)

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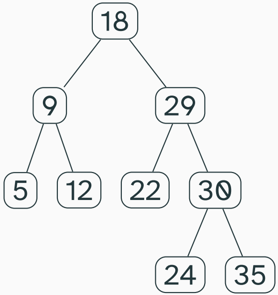

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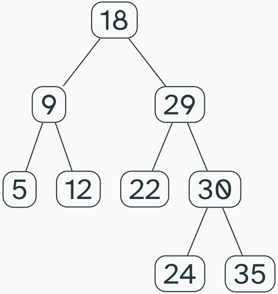
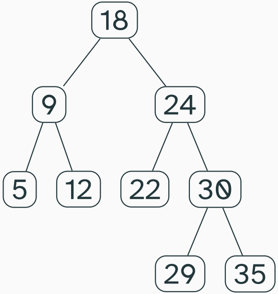


Is this a binary search tree?

No: note that *all* right descendants must be greater than the node

Finding an element in a binary search tree



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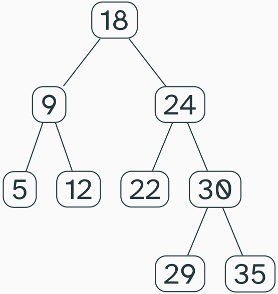

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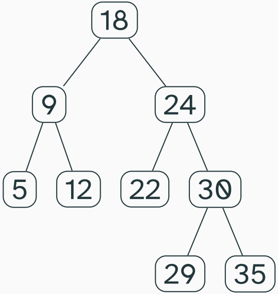

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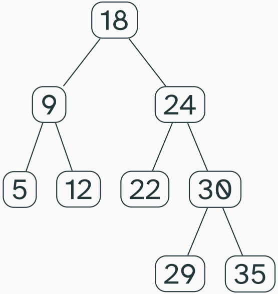

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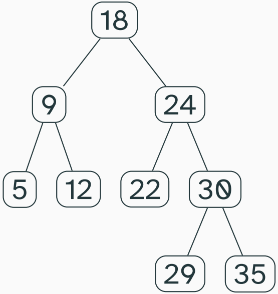

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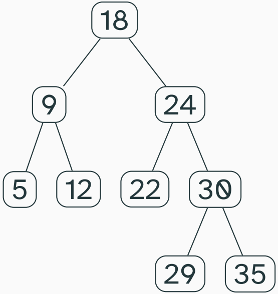

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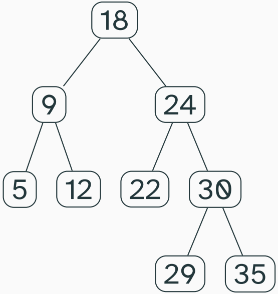

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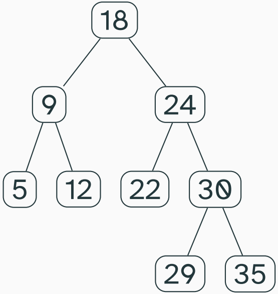

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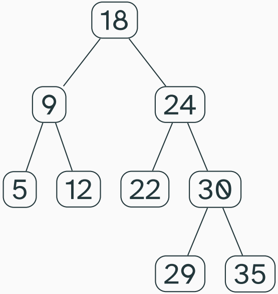

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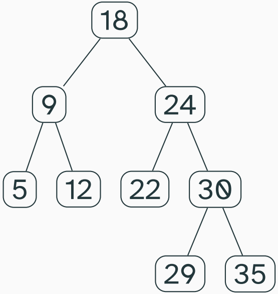

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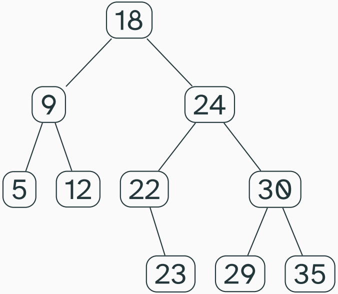

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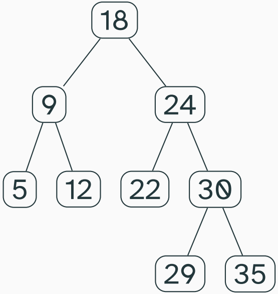

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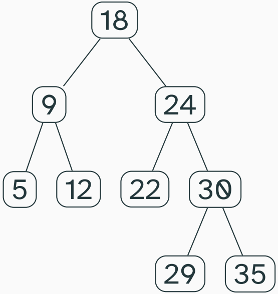

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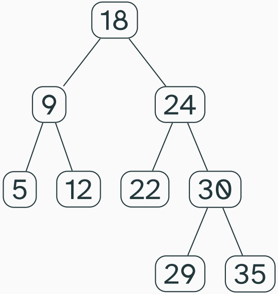

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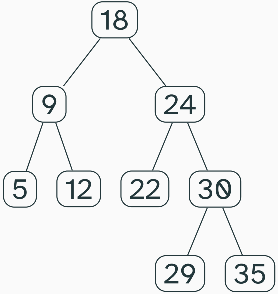

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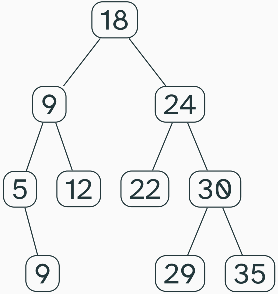

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Implementing a Binary Search Tree

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  - The `structure5` `BinarySearchTree<E>` assumes comparable items, but also allows a `Comparator` to be used...how?

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- Now: let's look at the code to locate an item, or to add it to the tree

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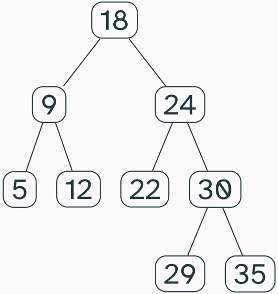

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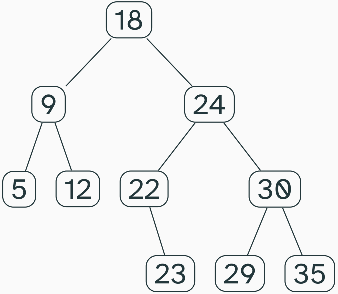

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