CSCI 136
Data Structures &
Advanced Programming

Lecture 26

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Profs Bill & Dan

Last Time

- Heaps
  - Implementation details
    - Data stored in an implicit binary tree in a Vector
  - Code inspection (`structure5.VectorHeap`)
  - Big-O of key operations

Today

- Heaps (again!)
  - Finish Implementation details
  - Some analysis + proofs
- Heapsort

Implementing Heaps: Recap

- Strategy: perform tree modifications that always preserve tree *completeness*, but may violate heap property. Then fix.
  - Add/remove never add gaps to array
    - We always add in next available array slot (left-most available spot in binary tree)
    - We always remove using "final" leaf
  - When elements are added and removed, do small amount of work to "re-heapify"
    - pushDownRoot(): recursively swaps large element down the tree
    - percolateUp(): recursively swaps small element up the tree

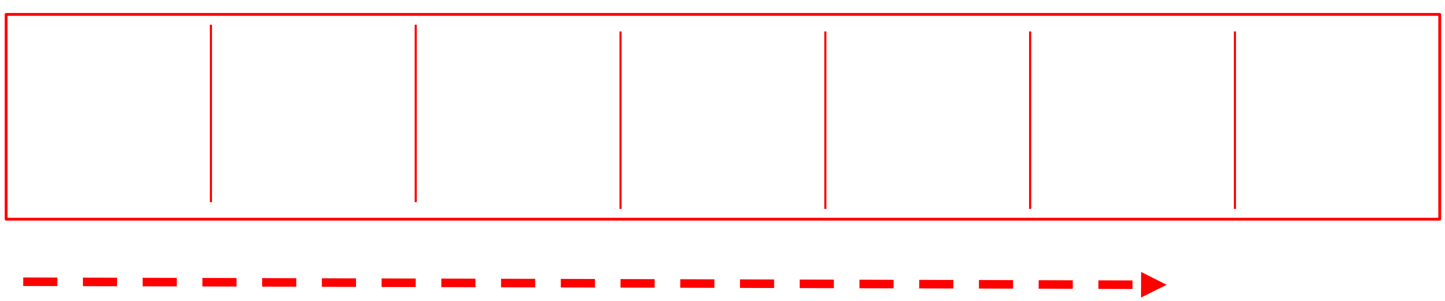
VectorHeap Summary

- Get is O(1), add/remove are both O(log n)
- Data is not *completely* sorted
  - A "partial" ordering is maintained for all root-to-leaf paths
- Note: `VectorHeap(Vector<E> v)`
  - Takes an unordered Vector and uses it to construct a heap
  - How?

Heapifying A Vector (or array)

Problem: You are given a Vector V that is not a valid heap, and you want to "heapify" V

- Method 1: Top-Down
  - Assume V[0...k] satisfies the heap property
  - Call `percolateUp` on item in location `k+1`
  - Now, `V[0..k+1]` satisfies the heap property



Grow valid heap region one element at a time

Practice Top-Down

Input:

- `int a[6] = {7,5,9,1,2,5,4}`

  0 1 2 3 4 5 6

```
for (int i = 0; i < a.length; i++)
    percolateUp(a, i);
```

Result: a is a valid heap!

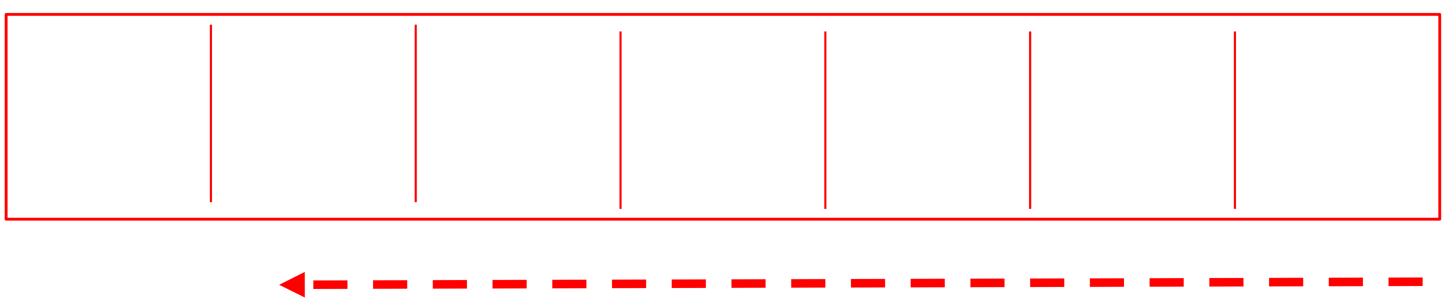
- `a = [1|2|4|7|5|9|5]`

  0 1 2 3 4 5 6

Heapifying A Vector (or array)

Problem: You are given a Vector V that is not a valid heap, and you want to "heapify" V

- Method II: Bottom-up
  - Assume V[k..n] satisfies the heap property
  - Now call pushDown on item in location k-1
  - Then V[k-1..n] satisfies heap property



Grow valid heap region one element at a time

Practice Bottom-Up

Input:

- `int a[6] = {7,5,9,1,2,5,4}`

  0 1 2 3 4 5 6

```
for (int i = a.length-1; i > 0; i--)
    pushDownRoot(a, i);
```

Result: a is a valid heap!

- `a = [1|2|4|5|7|5|9]`

  0 1 2 3 4 5 6

Let’s Compare

- Which is faster: Top down or Bottom Up?
  - Q: Think about a complete binary tree. Where do most of the nodes live?
  - A: The leaves!
  - Given that most of the nodes are leaves, should we percolateUp or pushDown?
    - To answer this, we should think about “how far” we need to move a node in the worst case.

Some Sums (for your toolbox)

$$\sum_{d=0}^{d=k} 2^d = 2^{k+1} - 1$$

$$\sum_{d=0}^{d=k} r^d = (r^{k+1} - 1)/(r - 1)$$

$$\sum_{d=1}^{d=k} d * 2^d = (k-1) * 2^{k+1} + 2$$

$$\sum_{d=1}^{d=k} (k-d) * 2^d = 2^{k+1} - k - 2$$

All of these can be proven by (weak) induction.

Try these proofs to hone your skills!

The second sum is called a geometric series. It works for any r≠0

Top-Down vs Bottom-Up

- **Top-down heapify (percolate up)**: elements at depth `d` may be swapped `d` times.
- The total # of swaps is:

(recall: h = log n)

$$\sum_{d=1}^{h} d2^d = (h-1)2^{h+1} = (\log n - 1)2n + 2$$

  - This is $O(n \log_2 n)$
  - Some intuition: most of the elements are in the lowest levels of the tree, so each of them might have to move to root: $O(\log_2 n)$ swaps per element

Top-Down vs Bottom-Up

- **Bottom-up heapify (push down):** elements at depth d may be swapped h−d times.
- The total # of swaps is:

$$\sum_{d=1}^{h}(h-d)2^d = 2^{h+1} - h - 2 = 2n - \log n + 2$$

  - This is O(n) — it beats top-down!
  - Some intuition: most of the elements are in the lowest levels of the tree, so each of them will only be pushed down (swapped) a small number of times

SO COOL!!!

HeapSort

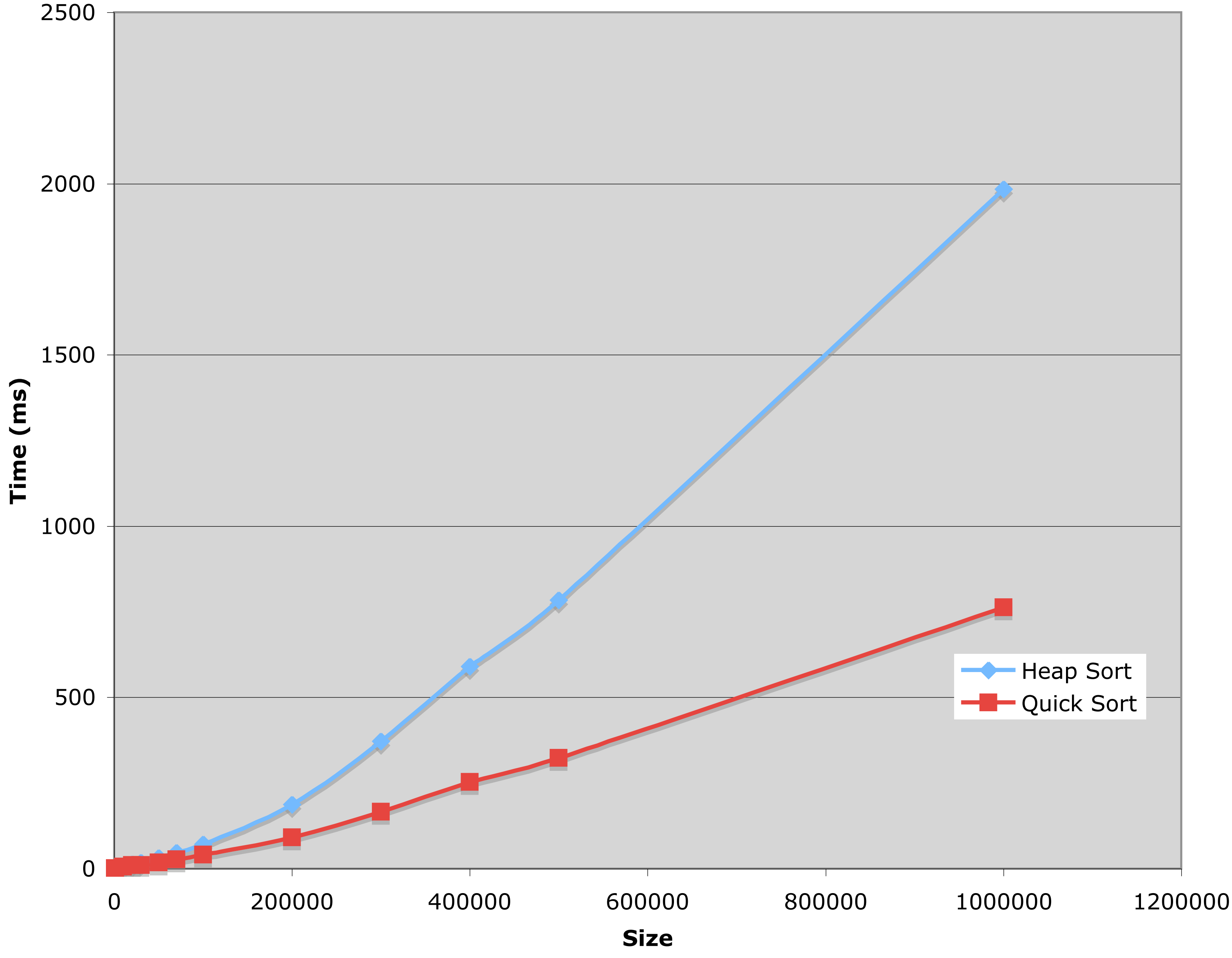
- Kind of an "Advanced" version of Selection Sort
- Strategy:
  1. Make a *max-heap*: array[0…n]
     - array[0] is largest value
     - array[n] is rightmost leaf
  2. Take the largest value (array[0]) and swap it with the rightmost leaf (array[n])
  3. Call pushDownRoot on array[0…n-1]
     - Now our "heap" is one element smaller, and the largest element is at end of array.

  Repeat until heap is empty and array is sorted

HeapSort

- Another O(n log n) sort method
- Heapsort is not *stable*
  - The relative ordering of elements is not preserved in the final sort
    - Why not?
      - There are multiple valid heaps given the same data
- Heapsort can be done *in-place*
  - No extra memory required!!!
  - Great for resource-constrained environments

Heap Sort vs QuickSort



Why Heapsort?

- Heapsort is slower than Quicksort in general
- Any benefits to heapsort?
  - *Guaranteed* O(n log n) runtime
- Works well on mostly sorted data, unlike quicksort
- Good for incremental sorting

More on Heaps

- Set-up: We want to build a *large* heap. We have several processors available.
- We'd like to use them to build smaller heaps and then merge them together
- Suppose we can share the array holding the elements among the processors.
  - How long to merge two heaps?
  - How complicated is it?
- What if we use BinaryTrees for our heaps?

Mergeable Heaps

- We now want to support the additional *destructive* operation merge(heap1, heap2)
- Basic idea: the heap with larger root somehow points into heap with smaller root
- Challenges
  - Points how? Where?
  - How much reheapifying is needed
  - How deep do trees get after many merges?

Skew Heap

- Heaps are not *necessarily* complete BTs
  - We made this requirement to guarantee performance in our VectorHeap representation
  - Rather than use Vector as underlying data structure, we can use a binary tree!
- Details are in the book, but at a high level…
  - The merge algorithm keeps the tree shallow over time
  - Theorem: Any set of `m` SkewHeap operations can be performed in `O(m log n)` time, where `n` is the total number of items in the SkewHeaps

Skew Heap: Merge Pseudocode

SkewHeap merge(SkewHeap S, SkewHeap T)

if either S or T is empty, return the other Case 1

if T.minValue < S.minValue

swap S and T *(S now has minValue)*

if S has no left subtree, T becomes its left subtree

Case 2

else

let temp point to right subtree of S

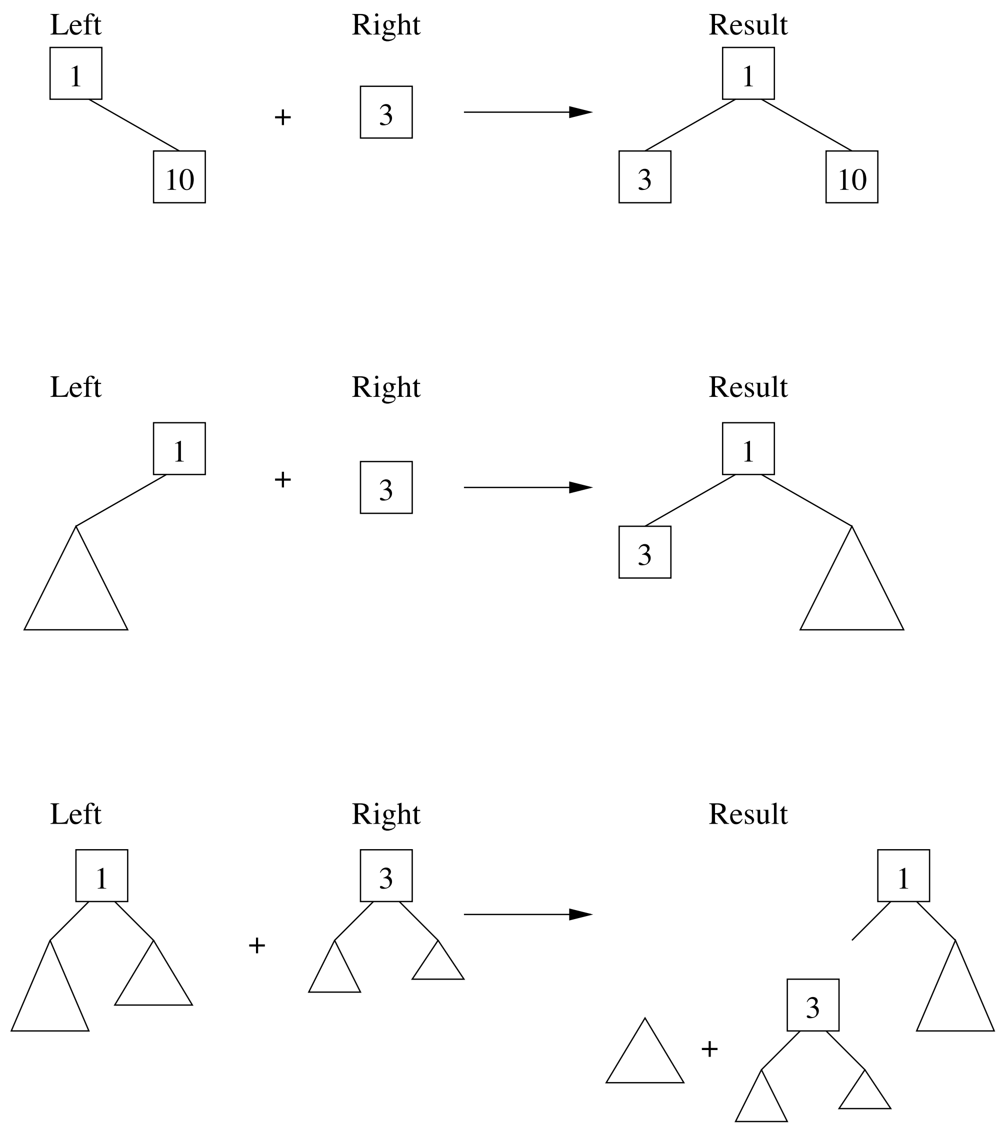
left subtree of S becomes right subtree of S

merge(temp, T) becomes left subtree of S

Case 3 (recurse)

return S

Skew Heap: Merge Examples



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Tree Summary

- Trees
  - Express hierarchical relationships
  - Level ordering captures the relationship
    - i.e., ancestry, game boards, decisions, etc.
- Heap
  - Partially ordered tree based on item priority
  - Node invariants: parent has higher priority than each child
  - Provides efficient PriorityQueue implementation