CSCI 136
Data Structures &
Advanced Programming

“Heapifying” an Array

Video Outline

- Heaps
  - Quick review of implementation strategies
  - Creating heaps from unsorted arrays
    - A top-down approach
    - A bottom-up approach
    - Some analysis + proofs

VectorHeap Design: Recap

- A heap is a *semi*-sorted tree
  - Rather than a "global" sort ordering, "partial" ordering is maintained for all root-to-leaf paths
- Data stored directly in an implicit binary tree
  - Children of `i` are at `2i+1` and `2i+2`
  - Parent is at `(i-1)/2`
- Tree is always complete
  - A prefix of the Vector is always occupied–no gaps

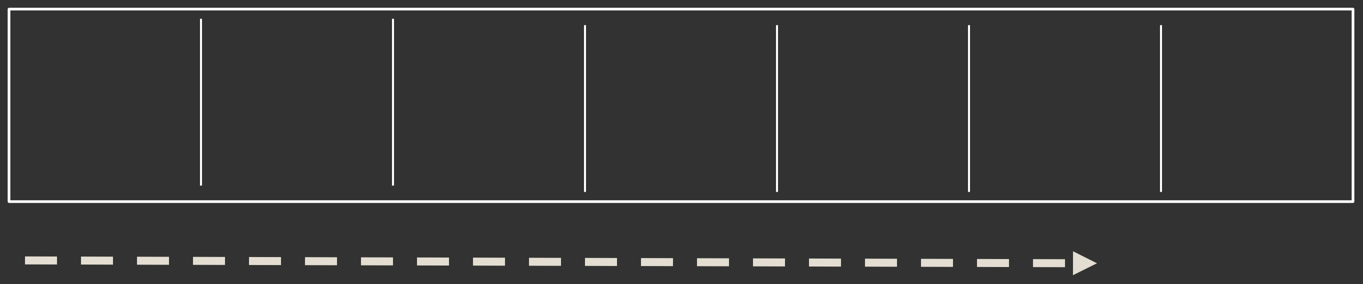
VectorHeap Operations: Recap

- Strategy: perform tree modifications that always preserve tree *completeness*, but may violate heap property. Then fix.
  - Add/remove never create gaps in between array elements
    - We always add in next available array slot (left-most available spot in binary tree)
    - We always remove using "final" leaf (rightmost element in array)
  - When elements are added and removed, do small amount of work to "re-heapify"
    - pushDownRoot(): recursively swaps large element down the tree
    - percolateUp(): recursively swaps small element up the tree

Heapifying A Vector (or array)

Problem: You are given a Vector `V` that is not a valid heap, and you want to "heapify" `V`

- Method I: **Top-Down**
  - Given `V[0...k]` satisfies the heap property
  - Call `percolateUp` on item in location `k+1`
  - Now, `V[0..k+1]` satisfies the heap property!



Grow valid heap region one element at a time

Practice Top-Down

```
int a[7] = {7,5,9,1,2,5,4};
for (int i = 0; i < a.length; i++)
    percolateUp(a, i);
```

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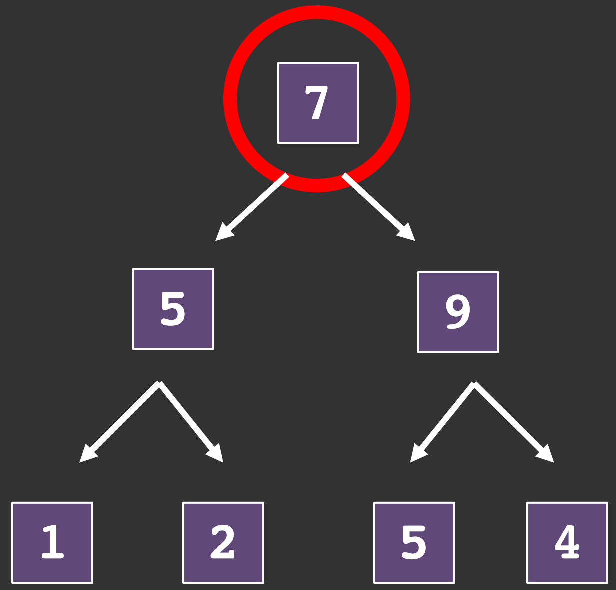

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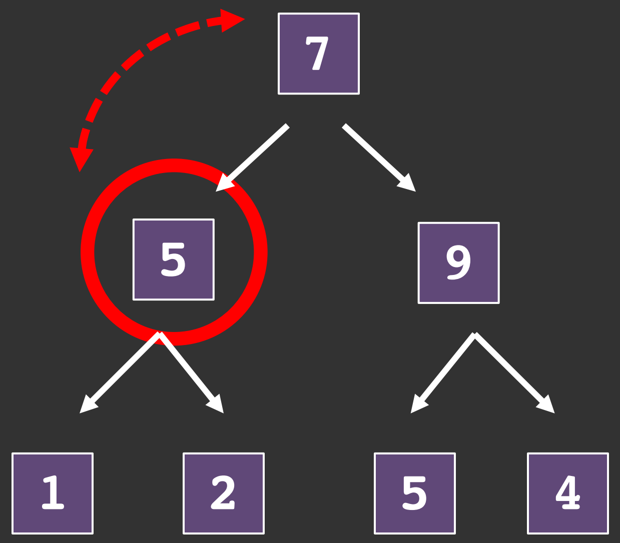

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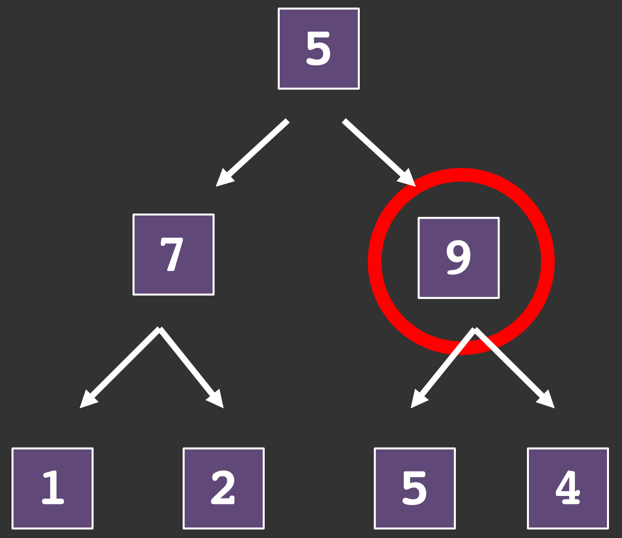

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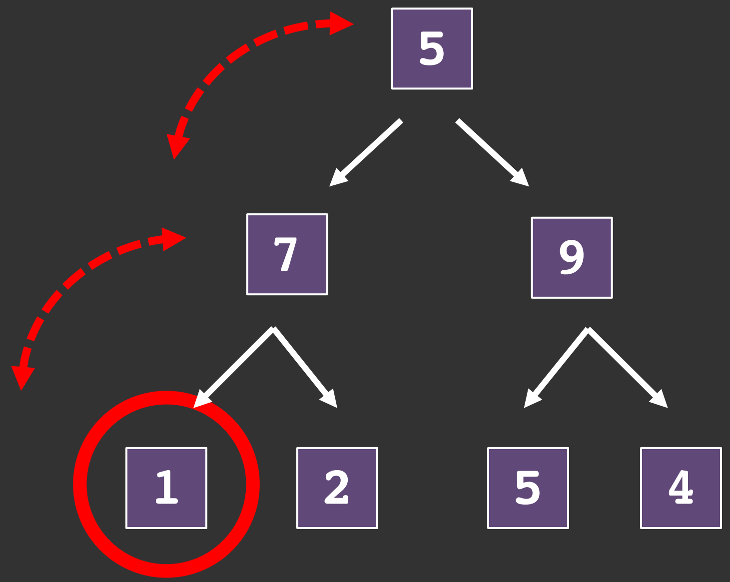

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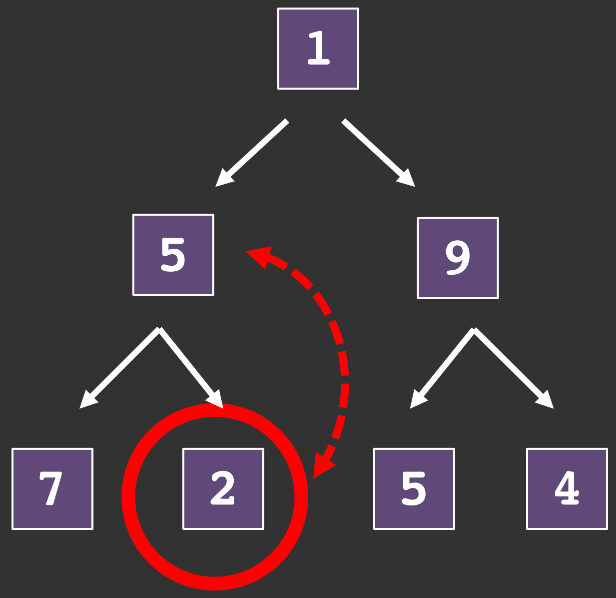

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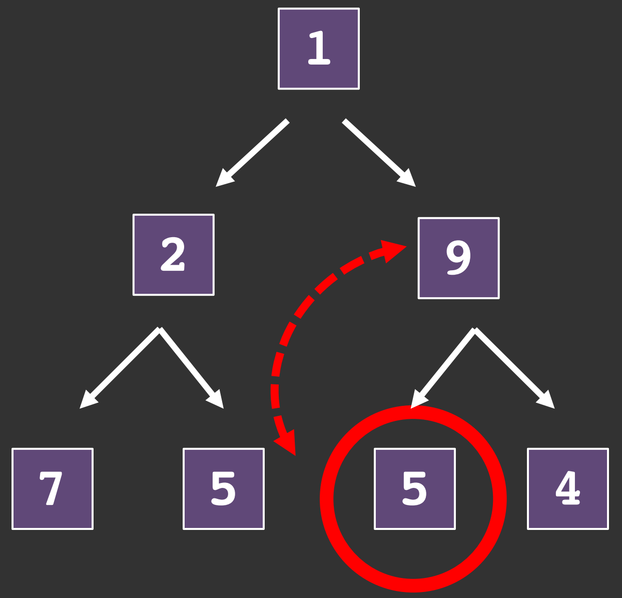

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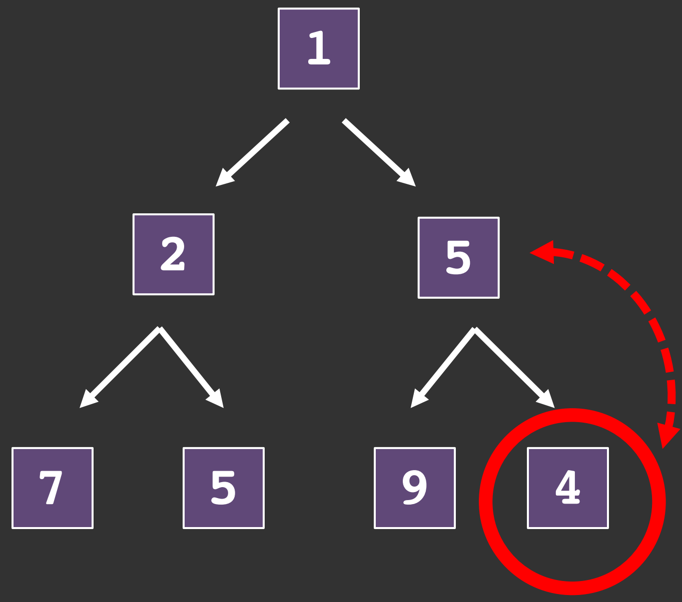
```



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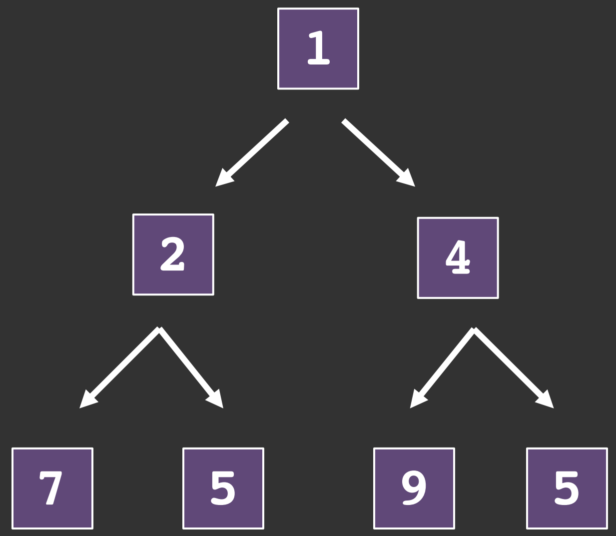

Heapifying A Vector (or array)

Problem: You are given a Vector `V` that is not a valid heap, and you want to "heapify" `V`

- Method II: Bottom-up
  - Given `V[k..n]` satisfies the heap property
  - Call pushDown on item in location `k-1`
  - Now, `V[k-1..n]` satisfies heap property!

Grow valid heap region one element at a time

Practice Bottom-up

```
int a[7] = {7,5,9,1,2,5,4};
for (int i = a.length-1; i > 0; i--)
    pushDownRoot(a, i);
```

```
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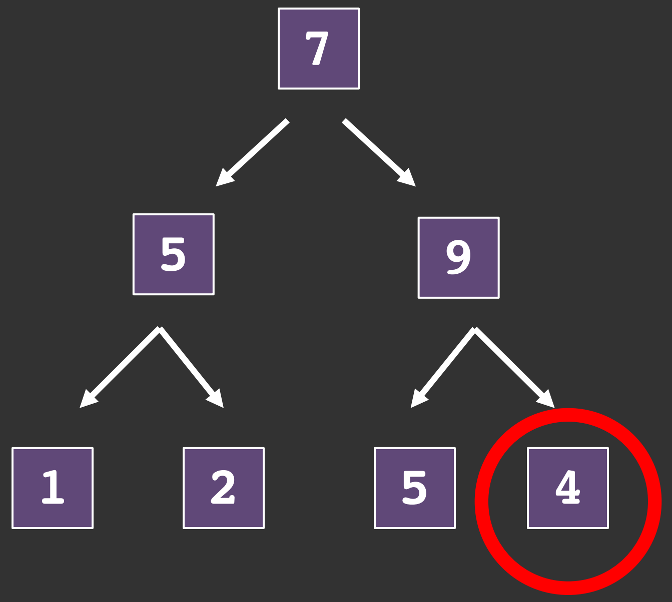

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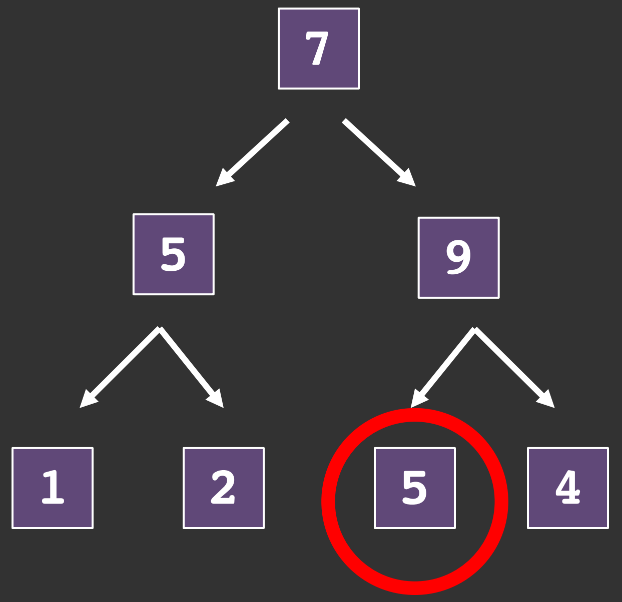

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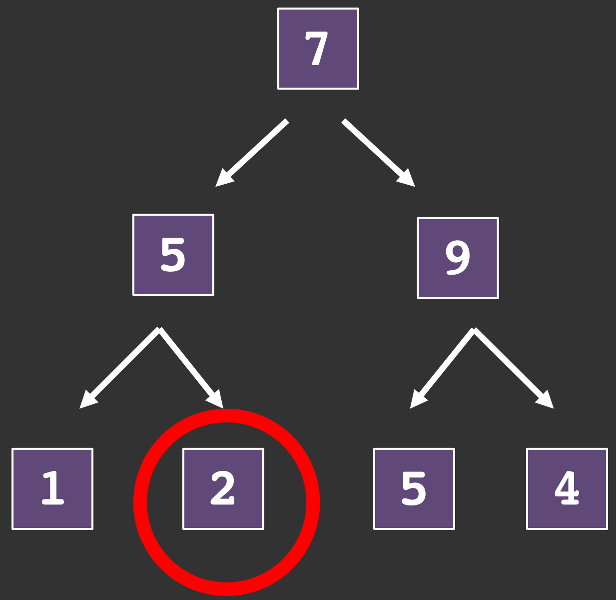

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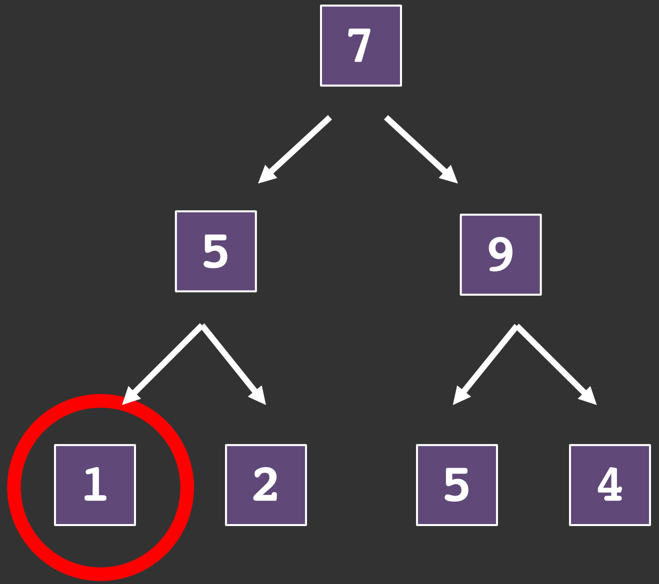

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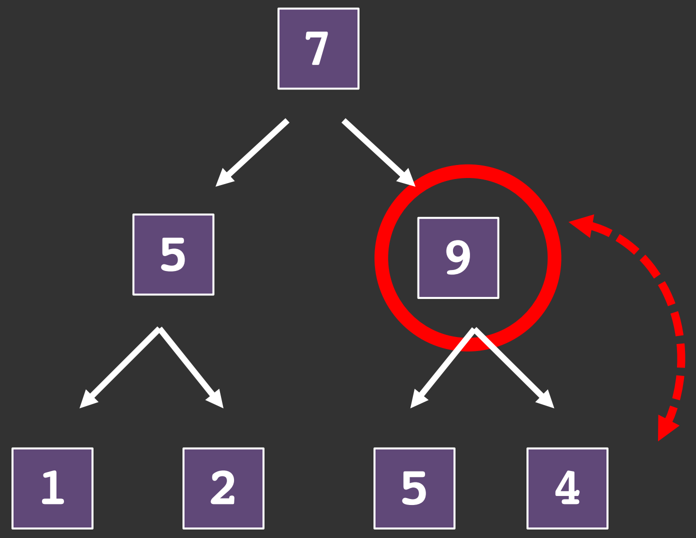

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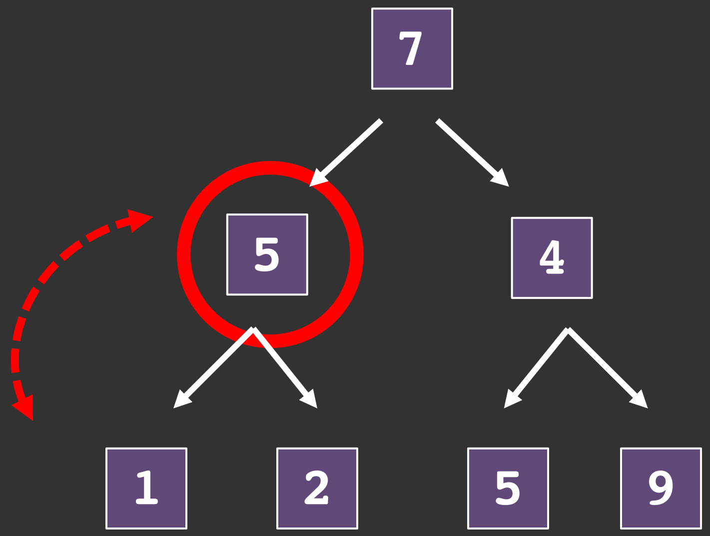
```



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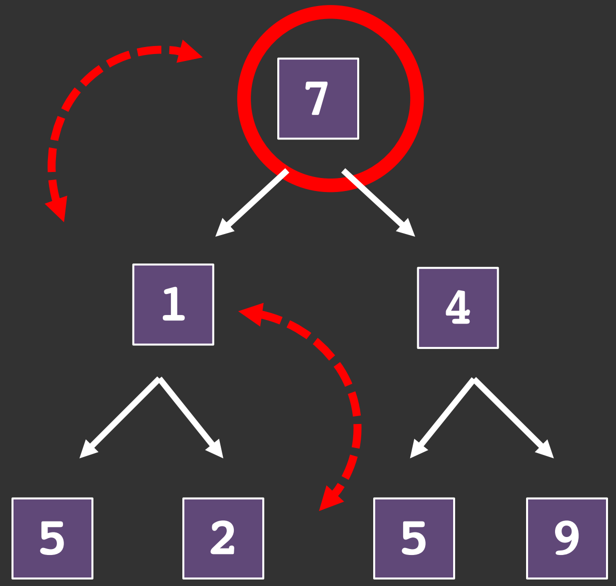

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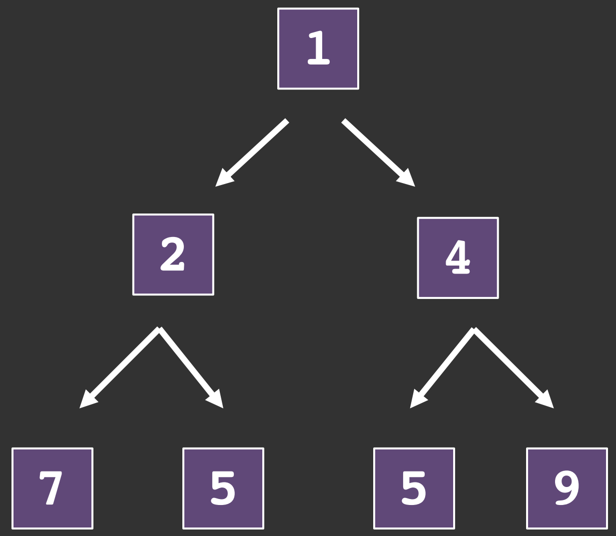
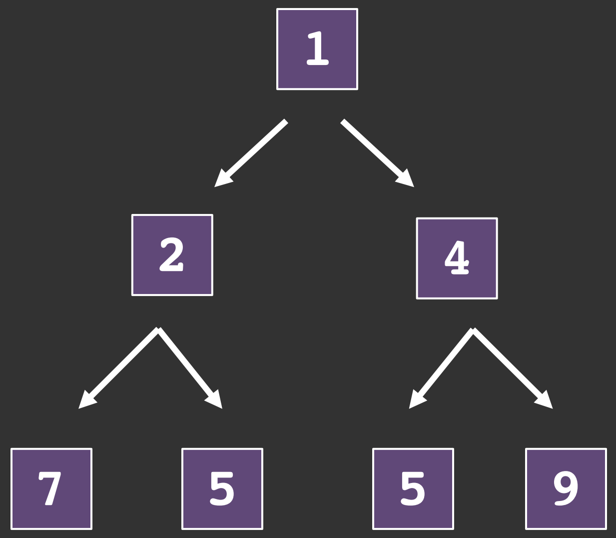


Let’s Compare



- Which is faster: Top down or Bottom Up?
  - Q: Think about a complete binary tree. Where do most of the nodes live?
  - A: The leaves!
  - Given that most of the nodes are leaves, should we `percolateUp` or `pushDown`?
- Let’s analyze this more formally

Some Sums (for your toolbox)

$$\sum_{d=0}^{n} 2^d = 2^{n+1} - 1$$

$$\sum_{d=1}^{n} (d)(2^d) = (n-1)(2^{n+1}) + 2$$

$$\sum_{d=1}^{n} (n-d)(2^d) = 2^{n+1} - 2n - 2$$

Both of these can be proven by (weak) induction.

Try these proofs to hone your skills!

(recall: h = log n)

Top-Down vs Bottom-Up

- Top-down heapify (percolate up): elements at depth d may be swapped d times.
- The total # of swaps is:

$$\sum_{d=1}^{h} d2^d = (h-1)2^{h+1} = (\log n - 1)2n + 2$$

- This is $O(n\log_2 n)$
- Some intuition: most of the elements are in the lowest levels of the tree, so each of them might have to move to root: $O(\log_2 n)$ swaps per element

Top-Down vs Bottom-Up

(recall: h = log n)

- Bottom-up heapify (push down): elements at depth d may be swapped h-d times.
- The total # of swaps is:

SO COOL!!!

$$\sum_{d=1}^{h}(h-d)2^d = 2^{h+1} - 2h - 2$$
$$= 2n - 2\log n - 2$$

- This is $O(n)$ — it beats top-down!
- Some intuition: most of the elements are in the lowest levels of the tree, so each of them will only be pushed down (swapped) a small number of times

Summary

- There are multiple valid ways to create a heap from an unsorted array
- The choices we make impact performance, so think carefully about the problem structure when developing your approach
- The same arguments apply to min-heaps and max-heaps: just inverse the swapping condition.

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Heapsort: an *in-place* $O(n \log n)$ sort

Video Outline

- Heapsort
  - Description
  - Comparison to Quicksort
  - When to use heapsort?

HeapSort

- How can we use a heap to sort?
- One idea (kind of like Selection Sort):
  - Heapify the array (max heap, with largest element at the root)
  - Keep removing the maximum element and putting it at the front

- This is the basic idea of heapsort. But how can we do this efficiently?

HeapSort

Strategy:

1. Make a *max-heap*: array[0…n]
   - array[0] is largest value
2. Take the largest value (array[0]) and swap it with the rightmost leaf (array[n])
3. Call pushDownRoot on array[0…n-1]

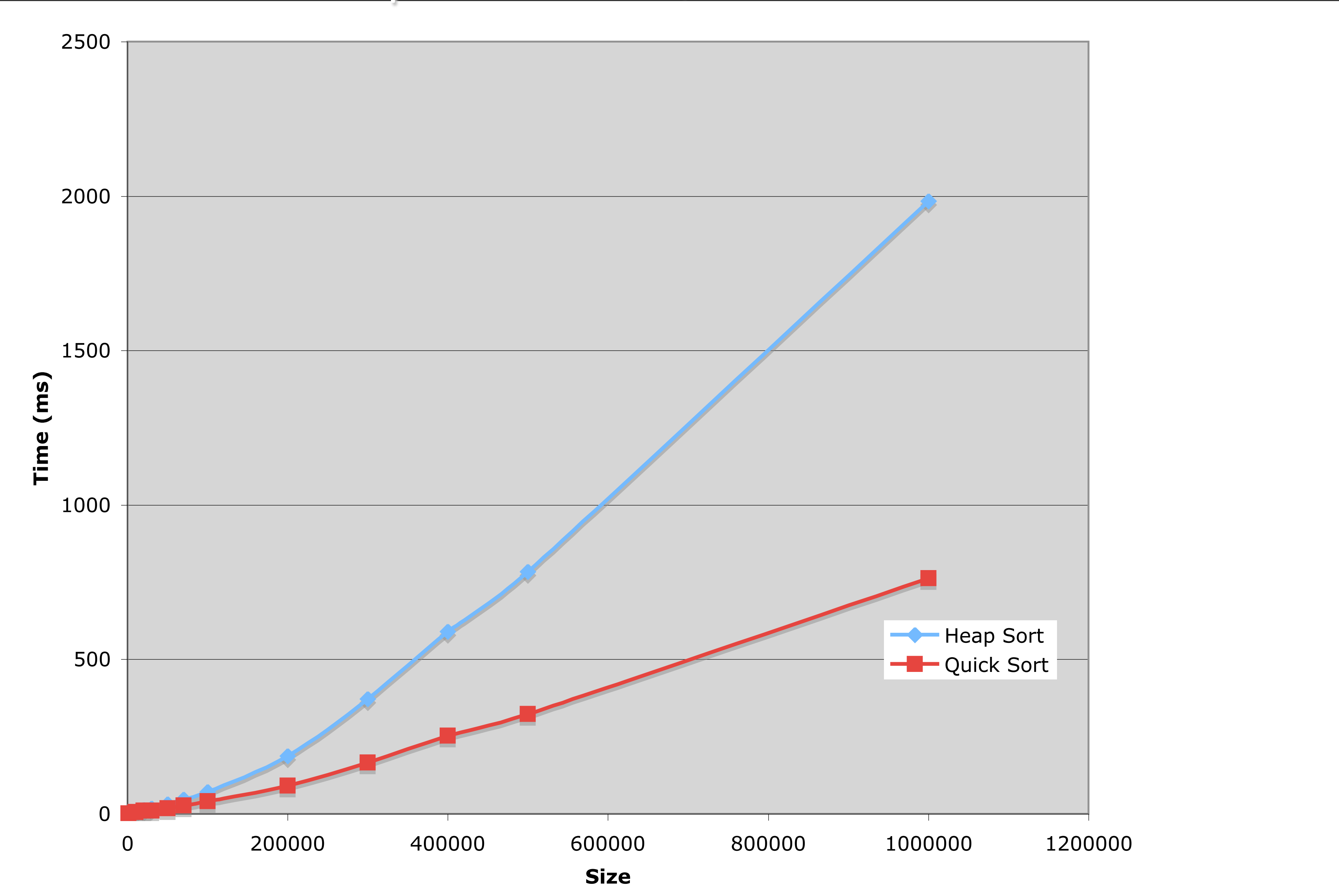
Now our “heap“ is one element smaller, and the largest element is at end of array.

Repeat until heap is empty and array is sorted

HeapSort

- Another O(n log n) sort method
- Heapsort is not *stable*
  - The relative ordering of elements is not preserved in the final sort
- Heapsort can be done *in-place*
  - No extra memory required!!!
  - Great for resource-constrained environments

Heap Sort vs QuickSort



Why Heapsort?

- Heapsort is slower than Quicksort in general
- Any benefits to heapsort?
  - *Guaranteed* O(n log n) runtime
  - In place
- Works well on mostly sorted data, unlike quicksort
- Good for incremental sorting

Heapsort Use Case

- Is “worst case” ever actually useful?
- Yes! Heapsort is used sometimes
- C++ standard library uses quicksort, but if quicksort is behaving badly it defaults to heapsort
- Also, (arguably) simpler than other efficient sorting algorithms

Heap Summary

- Heaps are a partially ordered tree based on item priority
  - Invariants: parent has higher priority than each child
- Heaps provide:
  - an efficient `PriorityQueue` implementation
  - an efficient building block for sorting (heapsort)
- We can efficiently manage heaps in an implicit array representation