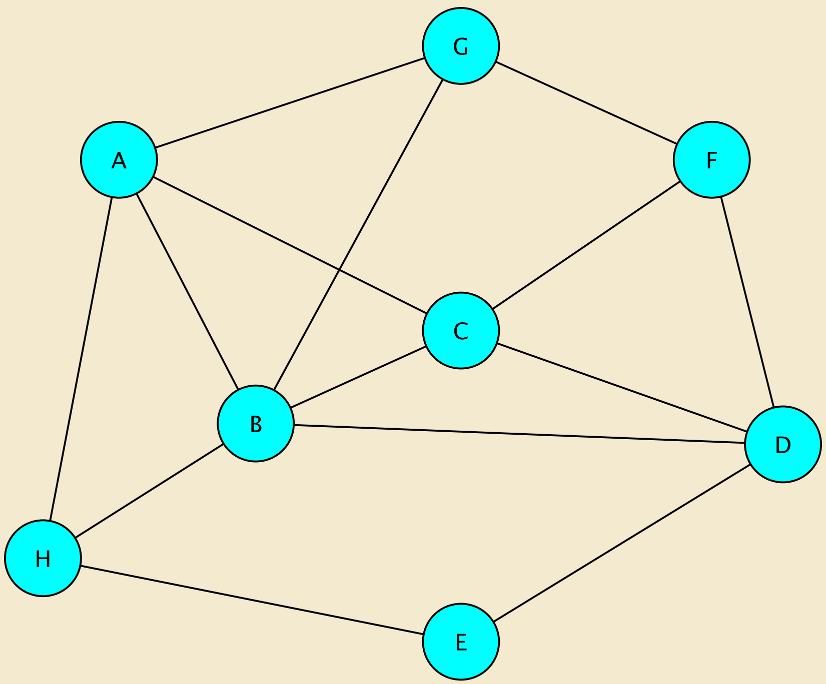
CSCI 136
Data Structures & Advanced Programming

Introduction to Graphs

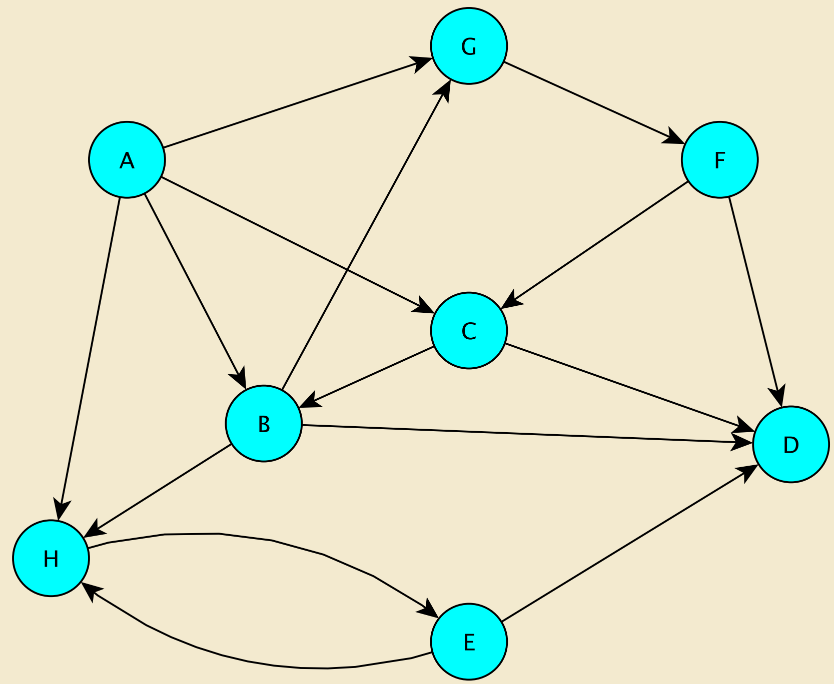
Graphs : Our Final Frontier

- Graphs as Mathematical Models
  - Basic Terminology
  - Important Structural Features
- Algorithms on Graphs
- Graph Data Structures
  - Undirected Graphs
  - Directed Graphs
- More Graph Algorithms

Basic Definitions & Concepts



An undirected graph

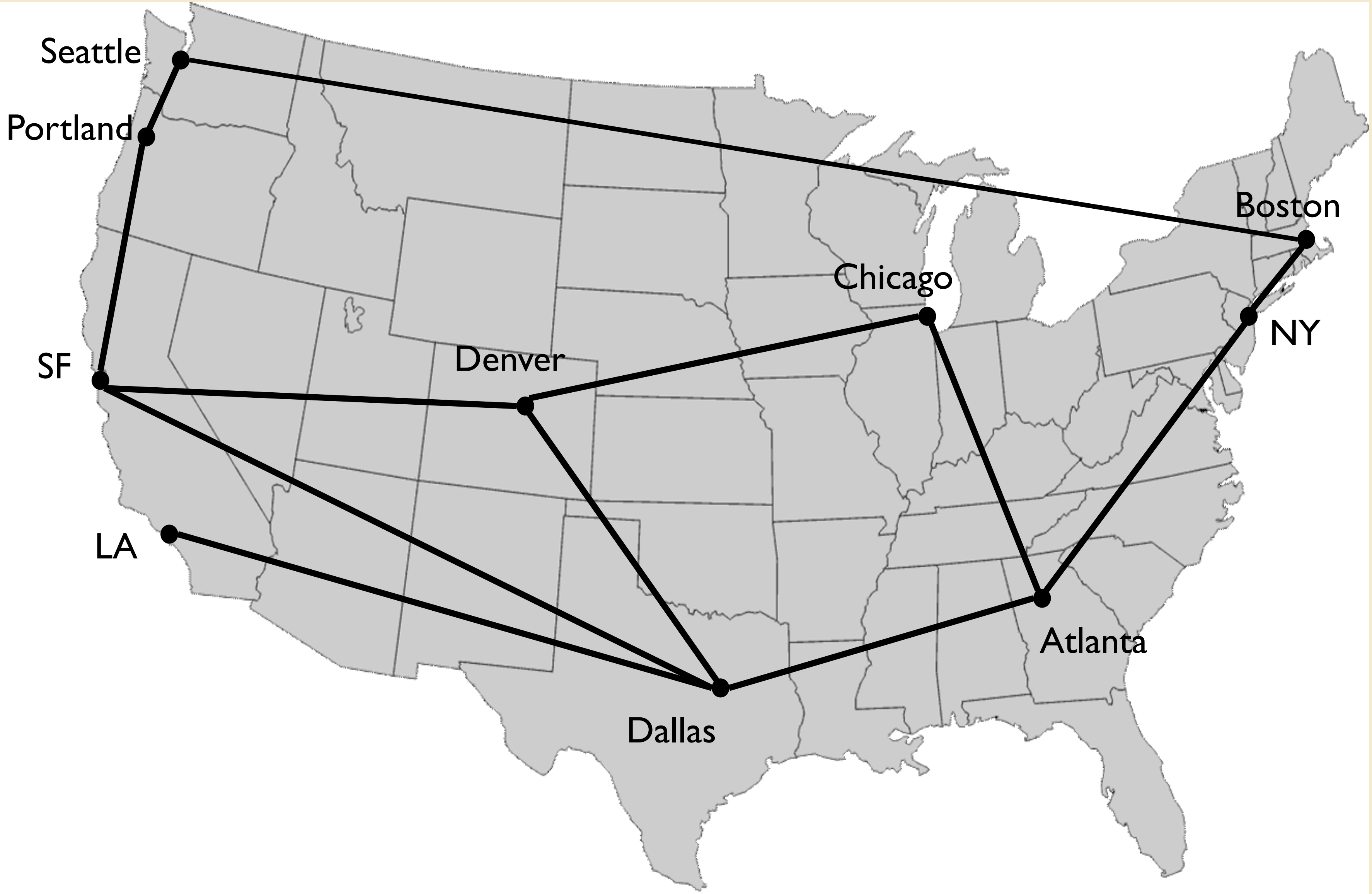


A directed graph

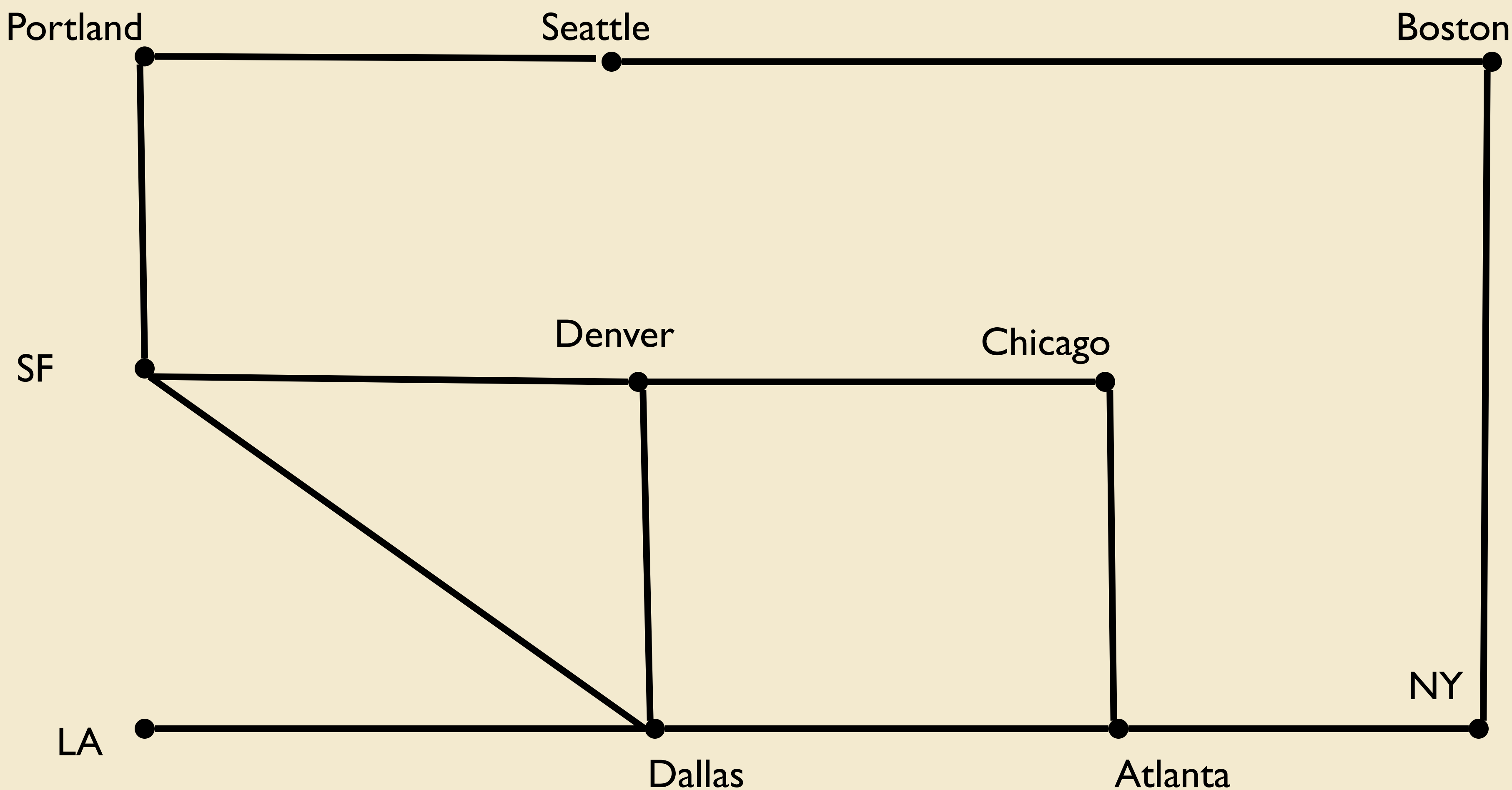
Graphs Describe the World

- Transportation Networks
- Communication Networks
- Molecular structures
- Dependency structures
- Scheduling
- Matching
- Graphics Modeling
- ....

Nodes = subway stops; Edges = track between stops

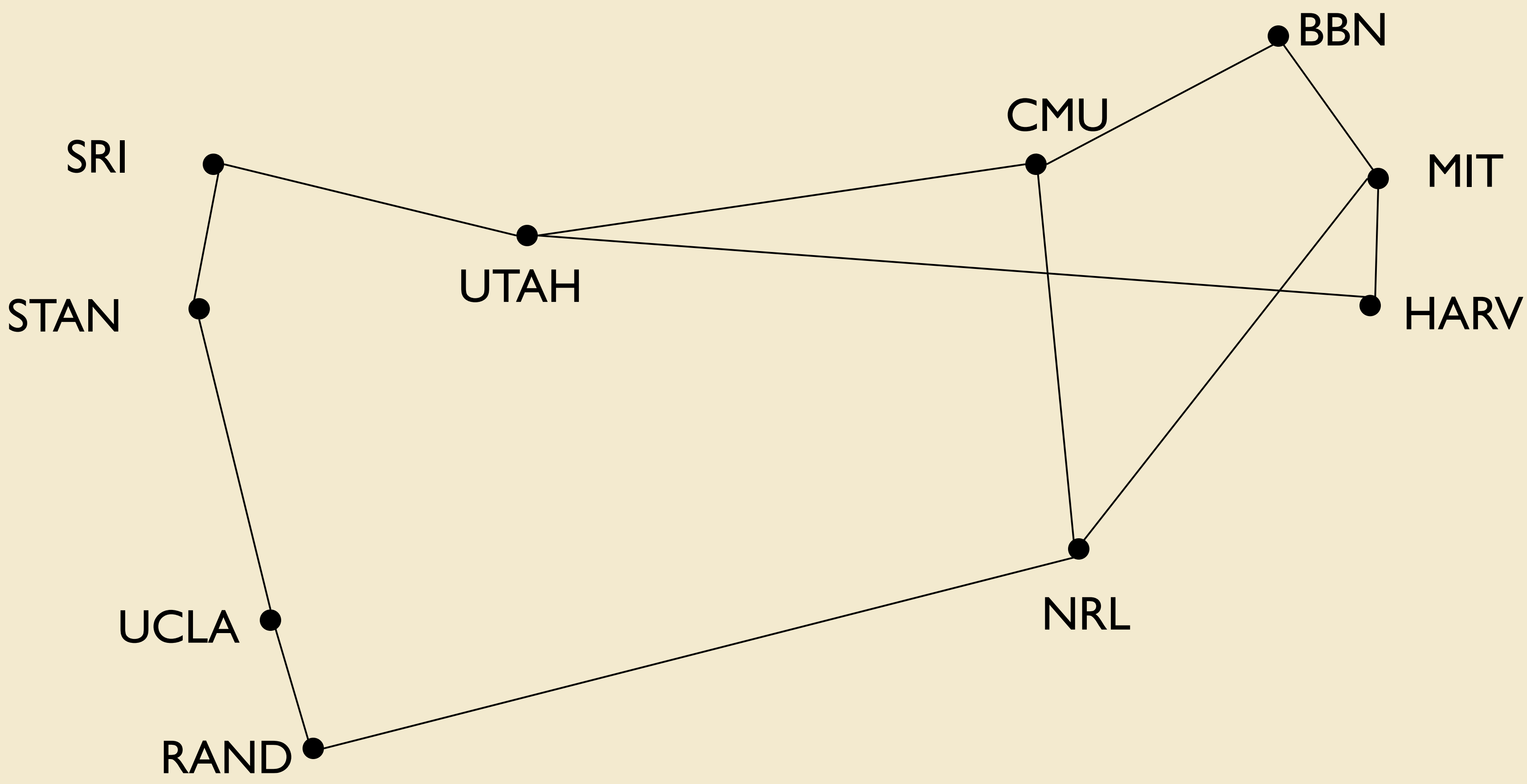


Nodes = cities; Edges = rail lines connecting cities

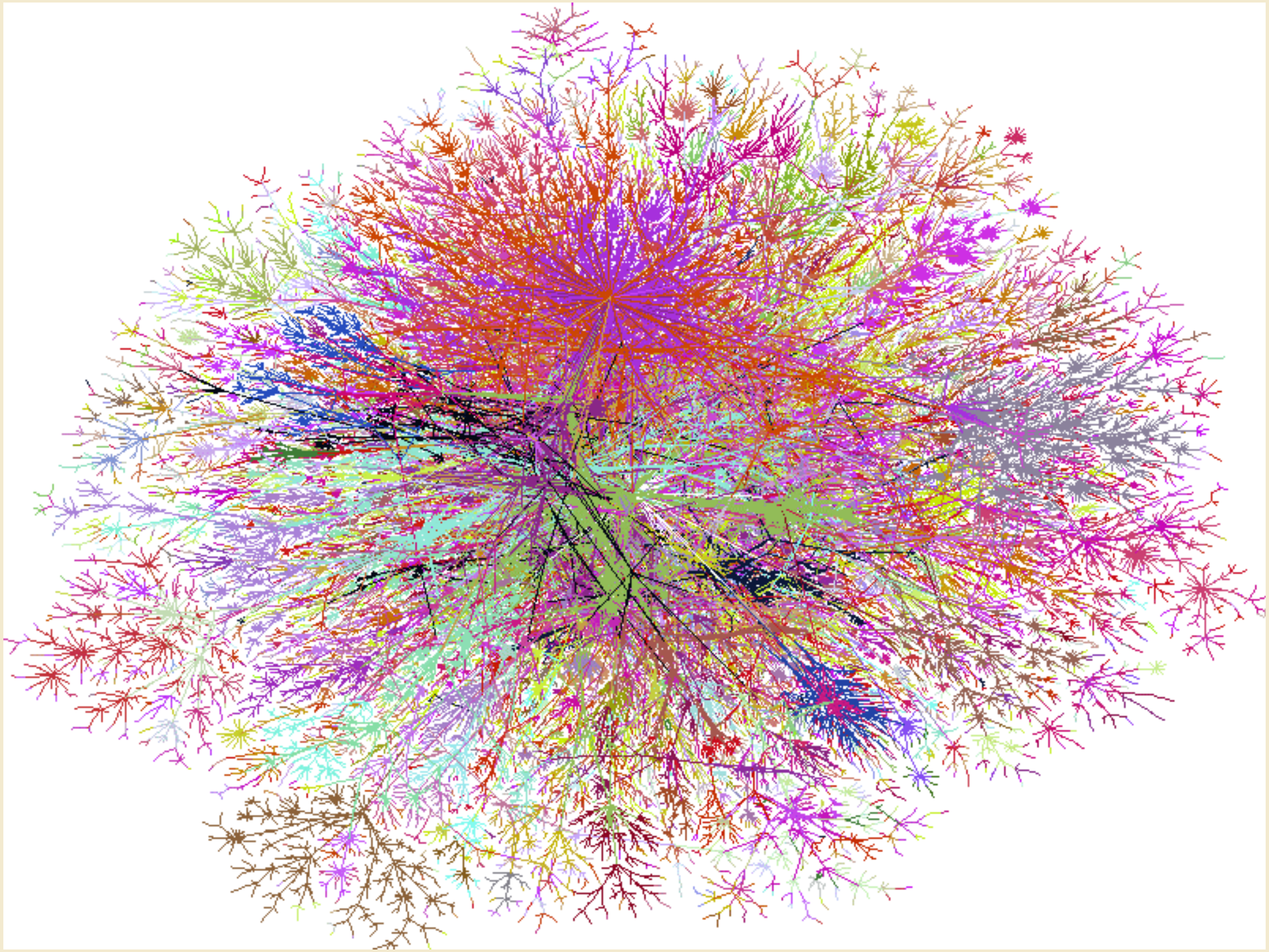


Note: Connections in graph matter, not precise locations of nodes

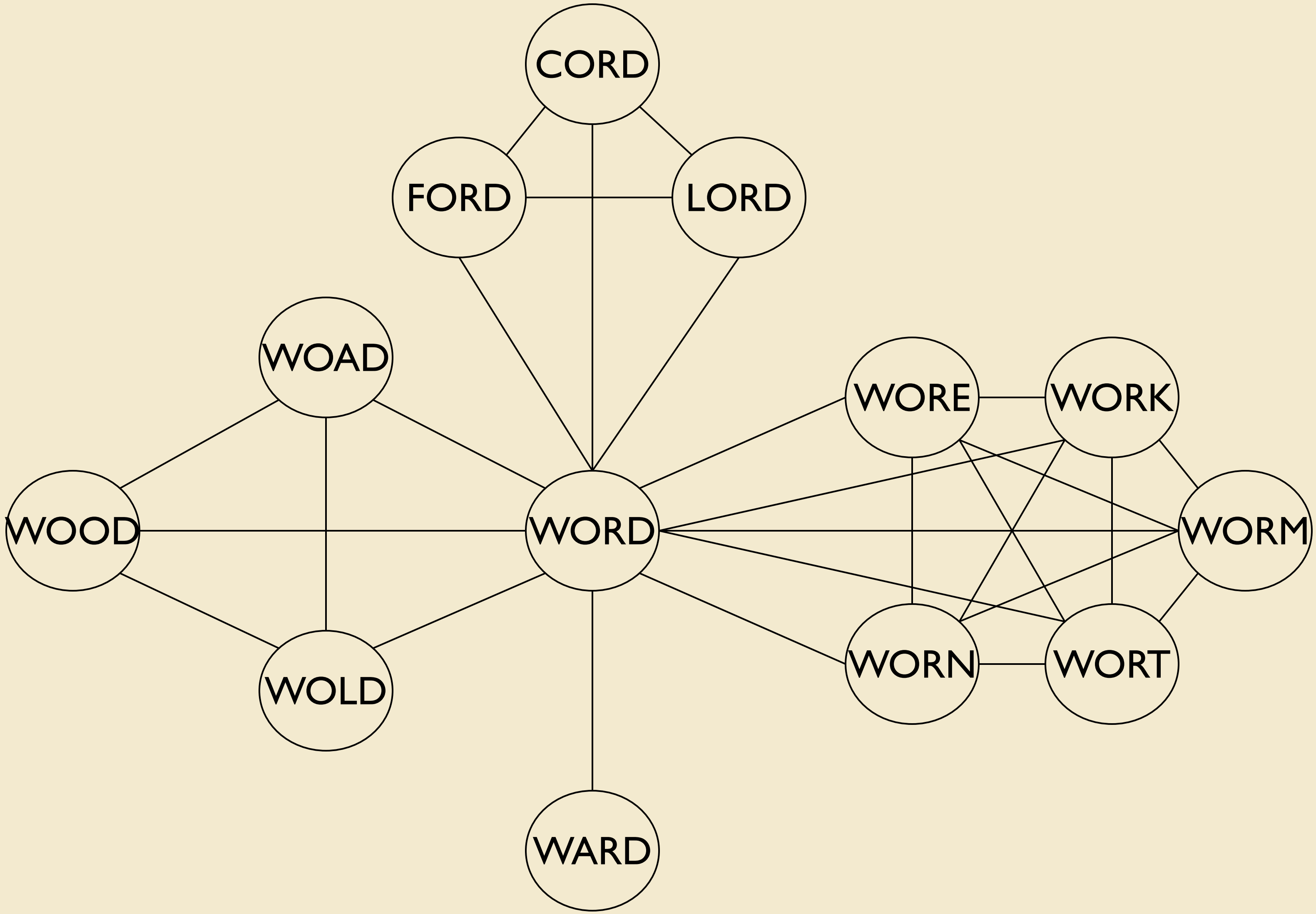
Internet (~1972)



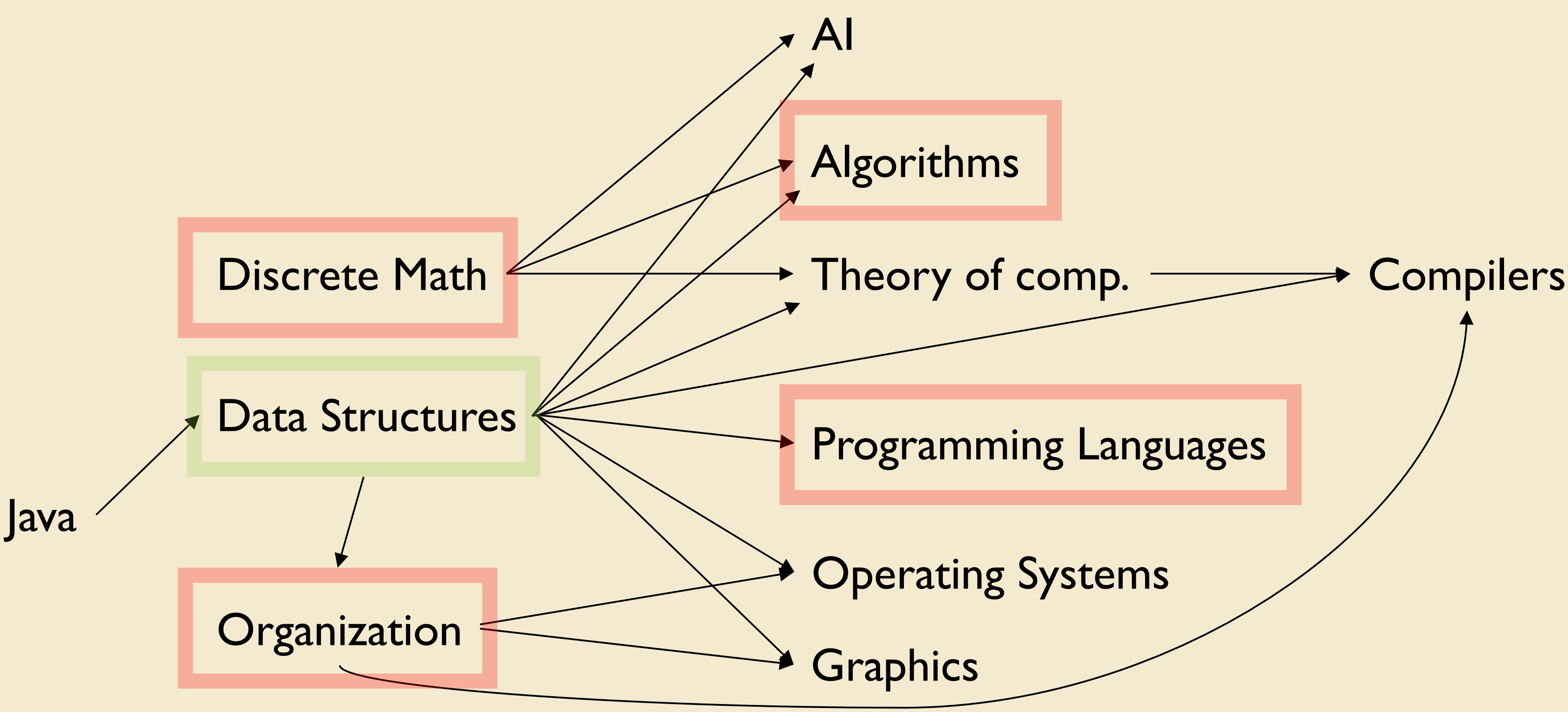
Internet (~1998)



Word Game

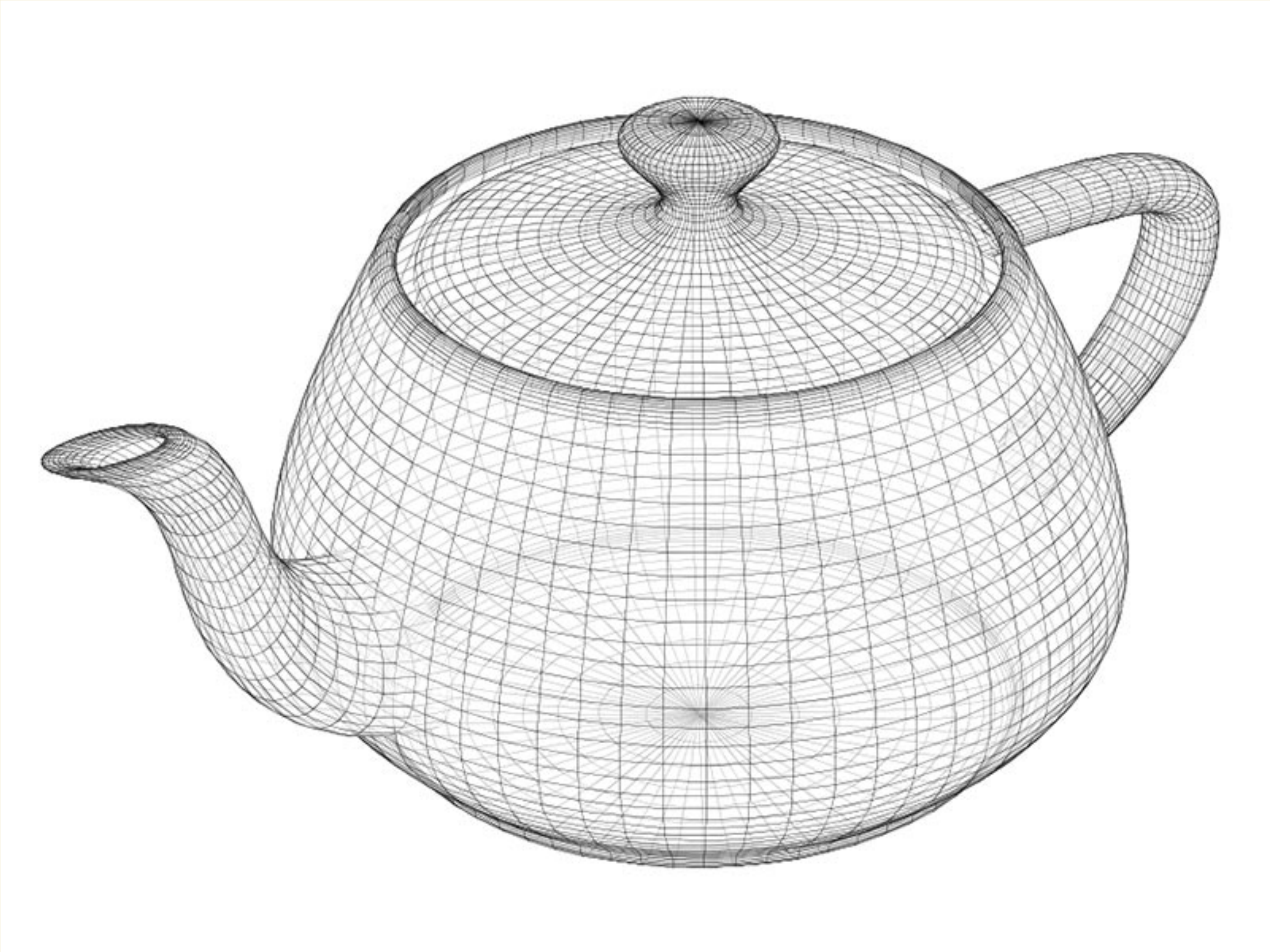

CORD
FORD
LORD
WOAD
WORE
WORK
WOOD
WORD
WORM
WORN
WORT
WOLD
WARD

CS Pre-requisite Structure (subset)

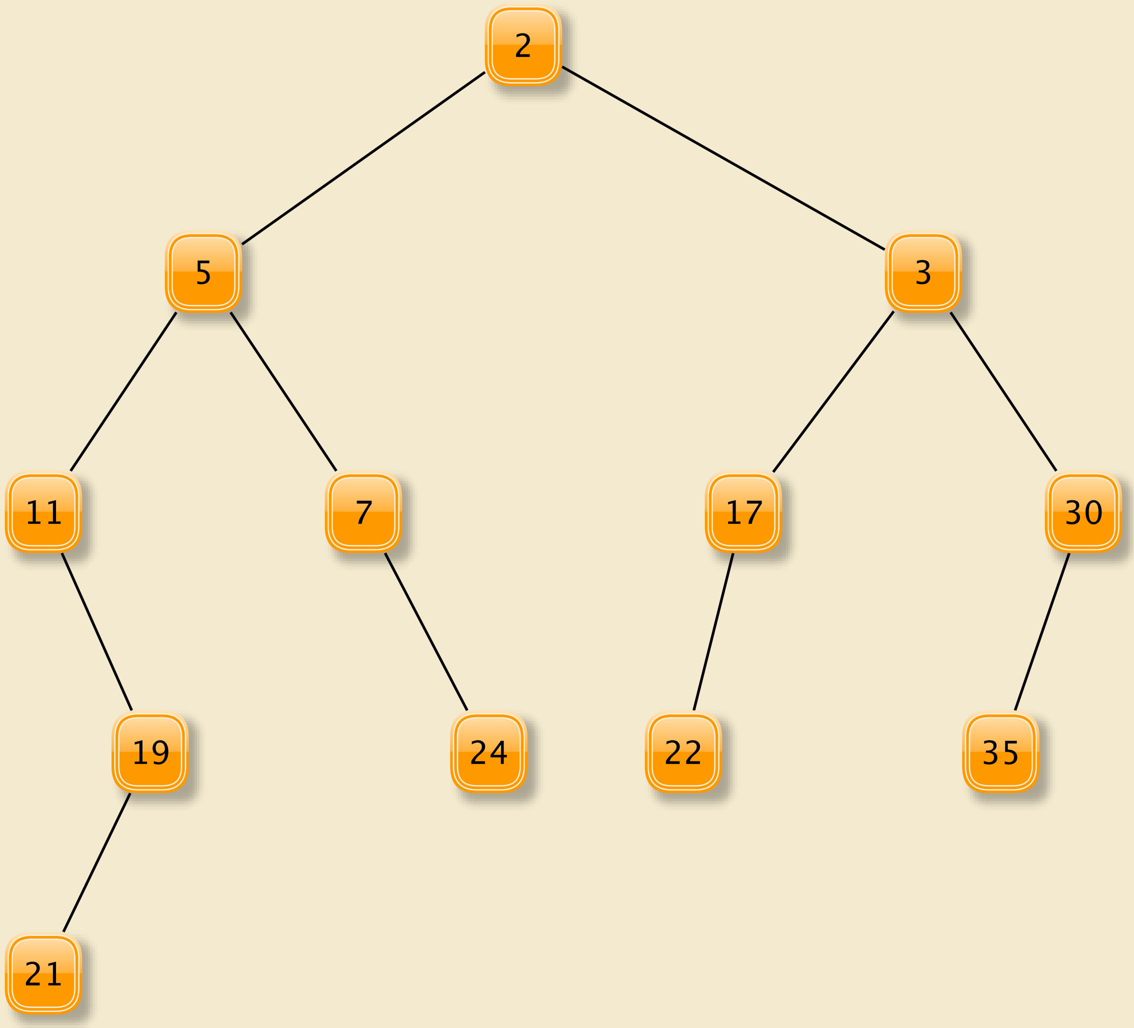


Nodes = courses; Edges = prerequisites ***

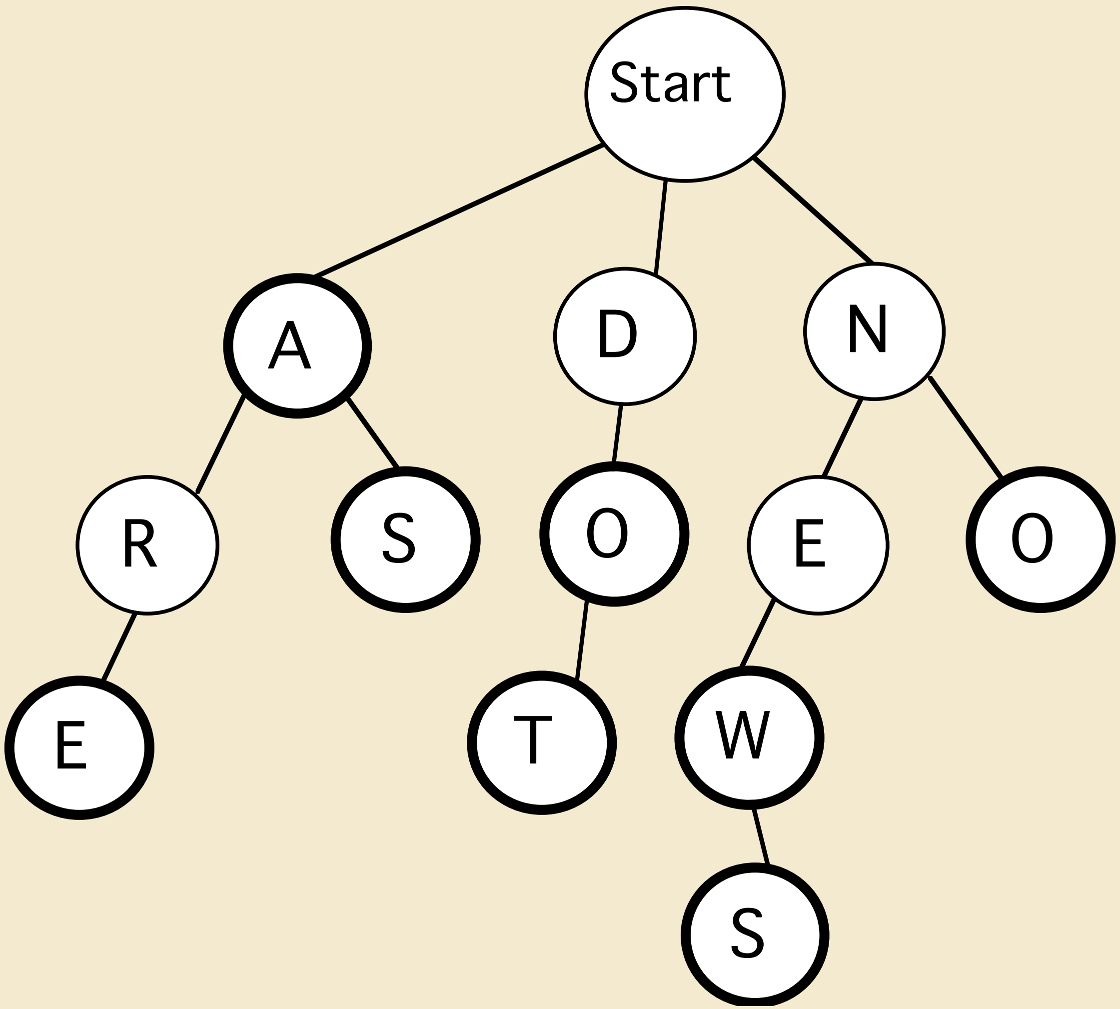
Wire-Frame Models



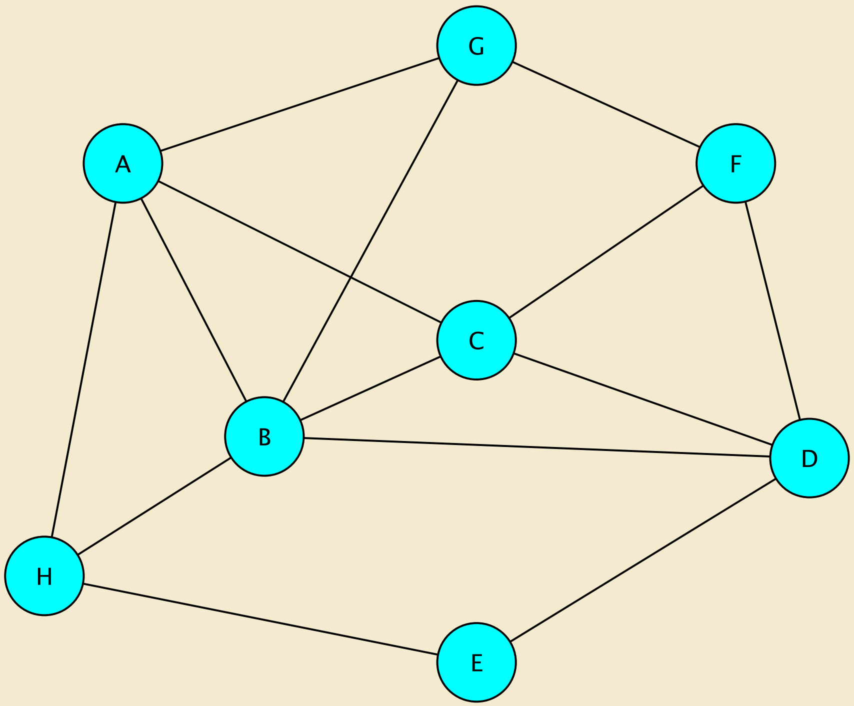
Priority Queue



Trie



Basic Definitions & Concepts



Definition:

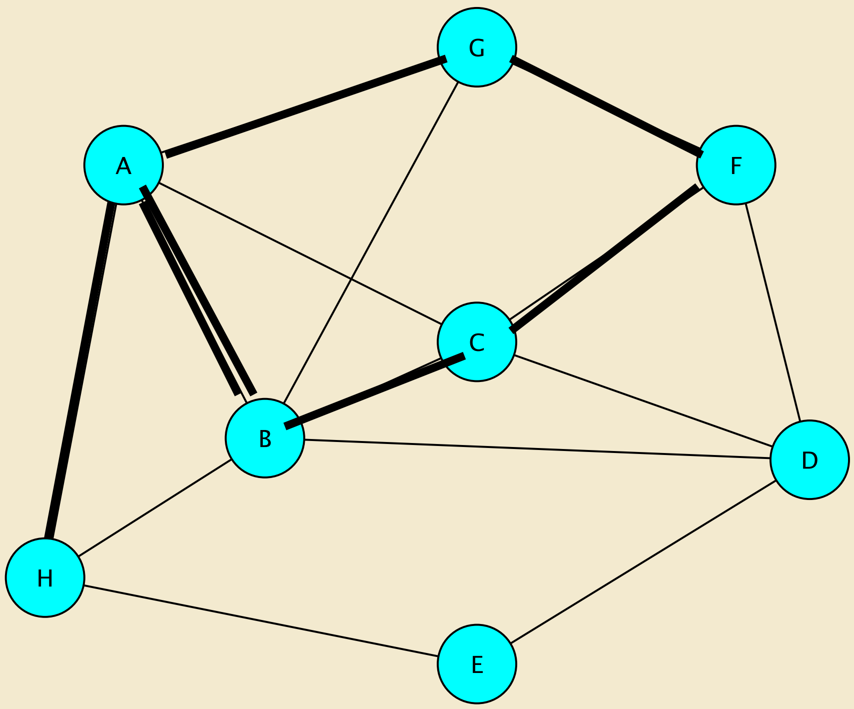
An *undirected graph* G = (V, E) consists of two sets

- V : the *vertices* of G
- E : the *edges* of G

Each edge e in E is defined by a *set* of two vertices: its *incident vertices*

- We write e = {u,v} and say that u and v are *adjacent*

Walking Around A Graph



Def'n: A *walk* from u to v in a graph G = (V,E) is an alternating sequence of vertices and edges

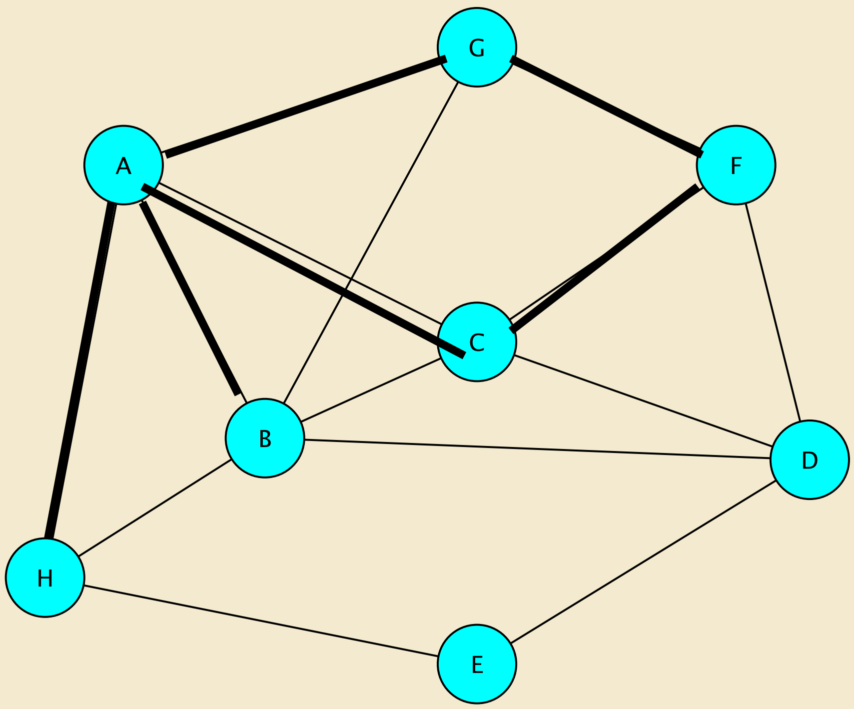
$u = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v$

such that each $e_i = \{v_{i-1}, v_i\}$ for $i = 1, \ldots, k$

- Note: A walk starts and ends with a vertex

B — A — G — F — C — B — A — H

Walking Around A Graph



Def'n: A *path* from u to v in a graph G = (V,E) is a walk that does not use any *edge* more than once

Def'n: A *simple path* is a path that does not use any *vertex* more than once

B — A — G — F — C — A — H

More Definitions : Walking In Circles

- A *closed walk* in a graph G = (V,E) is a walk

  $$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

  such that each $v_0 = v_k$
- A *circuit* is a path where $v_0 = v_k$
  - No repeated edges
- A *cycle* is a *simple path* where $v_0 = v_k$
  - No repeated vertices (uhm, except for v_0!)
- The length of any of these is the number of *edges* in the sequence

Little Tiny Theorems

- If there is a walk from u to v, then there is a walk from v to u.
- If there is a *walk* from u to v, then there is a *path* from u to v (and from v to u)
- If there is a *path* from u to v, then there is a *simple path* from u to v (and v to u)
- Every *circuit* through v contains a *cycle* through v
- Not every *closed walk* through v contains a *cycle* through v! [Try to find an example!]

Little Tiny Theorems

If there is a walk from u to v, then there is a walk from v to u.

Proof

- A walk from u to v is a sequence an alternating sequence of vertices and edges

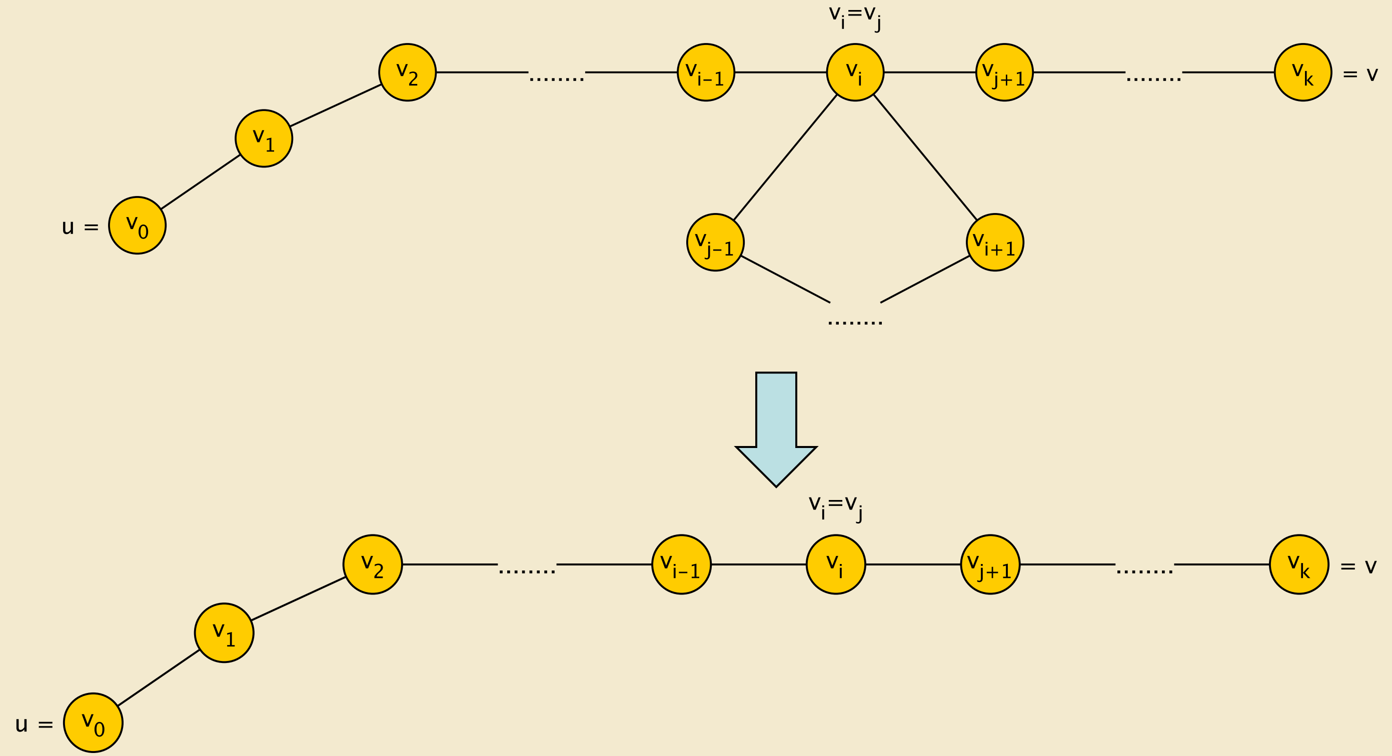
$$u = v_0, e_1, v_1, e_2, \ldots , v_{k-1}, e_k, v_k = v$$

- such that each $e_i = \{v_{i-1} , v_i\}$ for $i = 1, \ldots , k$
- But then $v = v_k, e_k, v_{k-1}, e_{k-1}, \ldots , v_1, e_1, v_0 = u$ is a walk from v to u.

Little Tiny Theorems

If there is a *path* from u to v, then there is a *simple path* from u to v.

Idea:



Little Tiny Theorems

Proof:

- Let $u = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v$ be a path from u to v (no edge appears twice)
- Suppose some v_i appears twice: that is, for some $j > i$, $v_j = v_i$. Then $e_{i+1} = \{v_i, v_{i+1}\}$ and $e_j = \{v_{j-1}, v_j\}$
- But $v_j = v_i$, so $e_j = \{v_{j-1}, v_i\}$ and so we can remove

$$e_{i+1}, v_{i+1}, e_{i+1}, \ldots, v_{j-1}, e_j$$

- from the original path obtaining the shorter path

$$u = v_0, e_1, v_1, \ldots, v_{k-1}, e_i, v_i = v_j, e_{j+1}, v_j, \ldots, e_k, v_k = v$$

- Repeat until no duplicate vertices remain.

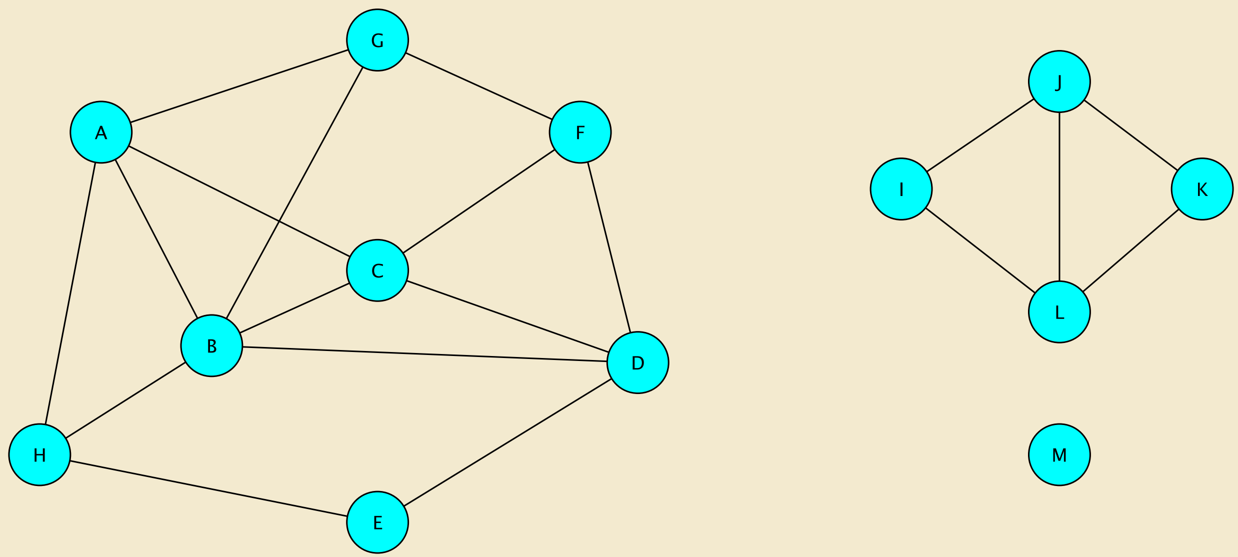
Another Useful Graph Fact

- If e = {u,v} we say e *is incident to u* (and to v)
- The *degree of* v is the number of edges incident to *v*
  - Denoted by *deg(v)*
- Thm: For any graph *G = (V,E)* : $\sum_{v \in V} \deg(v) = 2|E|$ where |*E*| is the number of edges in *G*
- Proof Hint: Induction on |*E*|: How does removing an edge change the equation?
  - Or: Count pairs *(v,e)* where *v* is incident with *e*

Reachability and Connectedness

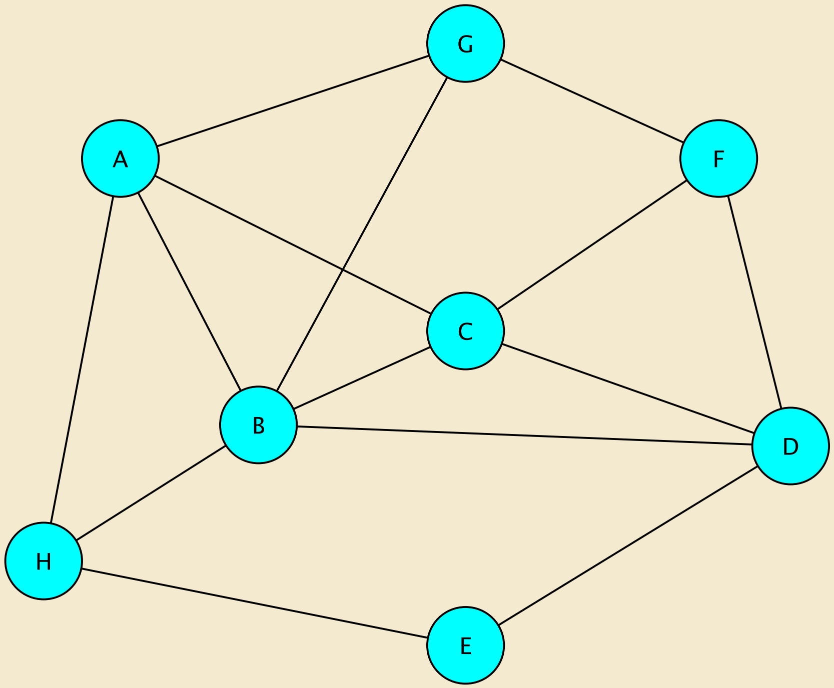
- Def'n: A vertex v in G is *reachable* from a vertex u in G if there is a path from u to v
- Note: v is reachable from u *if and only if* u is reachable from v
- Def'n: An undirected graph G is *connected* if for every pair of vertices u, v in G, v is reachable from u (and, of course, u from v)
- The set of all vertices reachable from v, along with all edges of G connecting any two of them, is called the *connected component of v*

Reachability and Connectedness



- 3 components
- A, B, C, D, E, F, G, H are all reachable from one another
  - As are I, J, K, L
  - M can reach only itself

Distance in Undirected Graphs

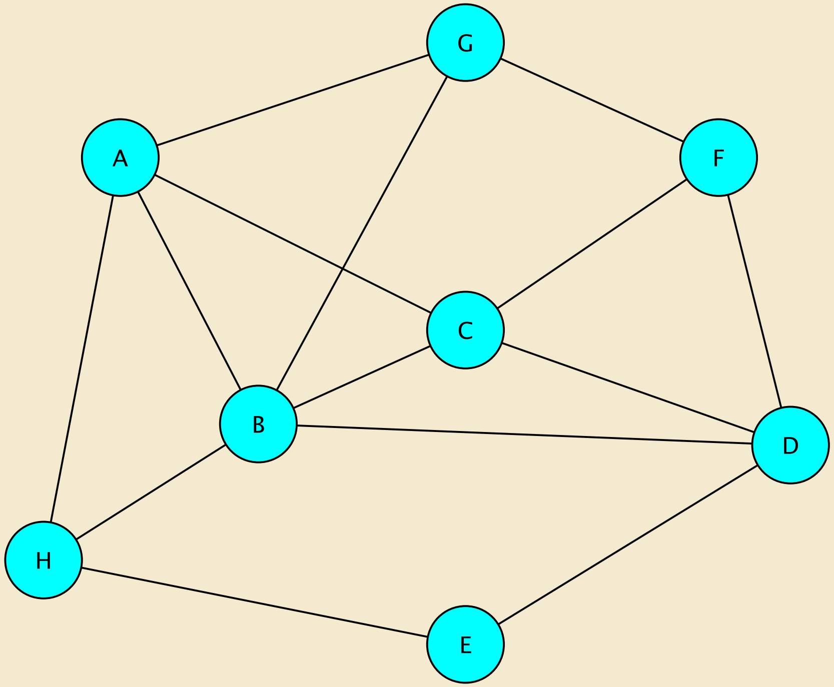


d(G,I) = 3
d(B,C) = 1
d(H,H) = 0

<u>Def'n</u>: The *distance* between two vertices *u* and *v* in an undirected graph *G=(V,E)* is the minimum of the path lengths over all *u-v* paths.

We write d(u,v)

Distance in Undirected Graphs



d(H,E) ≤ d(H,C) + d(C,E)
≤ 2 + 2 = 4
In fact, d(H,E) = 1

Distance satisfies

- $d(u,u) = 0$, for all $u \in V$
- $d(u,v) = d(v,u)$, for all $u,v \in V$
- $d(u,v) \leq d(u,w) + d(w,v)$, for all $u,v,w \in V$

This last property is called the *triangle inequality*

Algorithms on Graphs

- What are the basic operations we need to describe algorithms on graphs?
  - Given vertices u and v: are they *adjacent*?
  - Given vertex v and edge e, are they *incident*?
  - Given an edge e, get its incident vertices (*ends*)
  - How many vertices are adjacent to v? (*degree* of v)
    - The vertices adjacent to v are called its *neighbors*
  - Get a list of the vertices *adjacent* to v
    - From which we can get the edges *incident* with v

Basic Graph Algorithms

- We'll look at a number of graph algorithms
  - Connectedness: Is G connected?
    - If not, how many connected components does G have?
  - Cycle testing: Does G contain a cycle?
    - Does G contain a cycle through a given vertex?
  - If the edges of G have costs:
    - What is the cheapest connected subgraph of G that contains every vertex?
    - What is a cheapest path from u to v?
  - And more....

Testing Connectedness

- How can we determine whether G is connected?
  - Pick a vertex v; see if every vertex u is reachable from v
- How could we do this?
  - Visit the neighbors of v, then visit their neighbors, etc. See if you reach all vertices
    - Assume we can mark a vertex as "visited"
- How do we *efficiently* manage all this visiting?

Reachability: Breadth-First Search

```
BFS(G, v)        // Do a breadth-first search of G starting at v
// pre: all vertices are marked as unvisited
count ← 0;
Create empty queue Q; enqueue v; mark v as visited; count++
While Q isn't empty
        current ← Q.dequeue();
        for each unvisited neighbor u of current :
                add u to Q; mark u as visited; count++
return count;
```

Now compare value returned from BFS(G,v) to size of V

BFS Theorem

Thm. BFS(G,v) visits exactly those vertices u reachable from v.

Proof: We'll show that if u is reachable from v then BFS(G,v) visits u by induction on $d = d(v,u)$

- Base Case: $d = 0$. Then $u = v$.
  - v is reachable from v and BFS(G,v) visits v
- Induction Hypothesis: For some $d \geq 0$, if $d(u,v) = d$ then BFS(G,v) visits u.

BFS Theorem

- Induction Step: Assume now that $d(u,v) = d+1$
  - Let $v = v_0, e_1, v_1, e_2, v_2, \ldots, v_d, e_{d+1}, v_{d+1} = u$ be a path of length $d+1$ from v to u
  - Then $v = v_0, e_1, v_1, e_2, v_2, \ldots, v_d$ is a path of length d from v to v_d
  - By I.H., v_d is visited by BFS(G,v) and put in Q
  - So v_d will be dequeued and all of its unvisited neighbors, including u, will be marked as visited

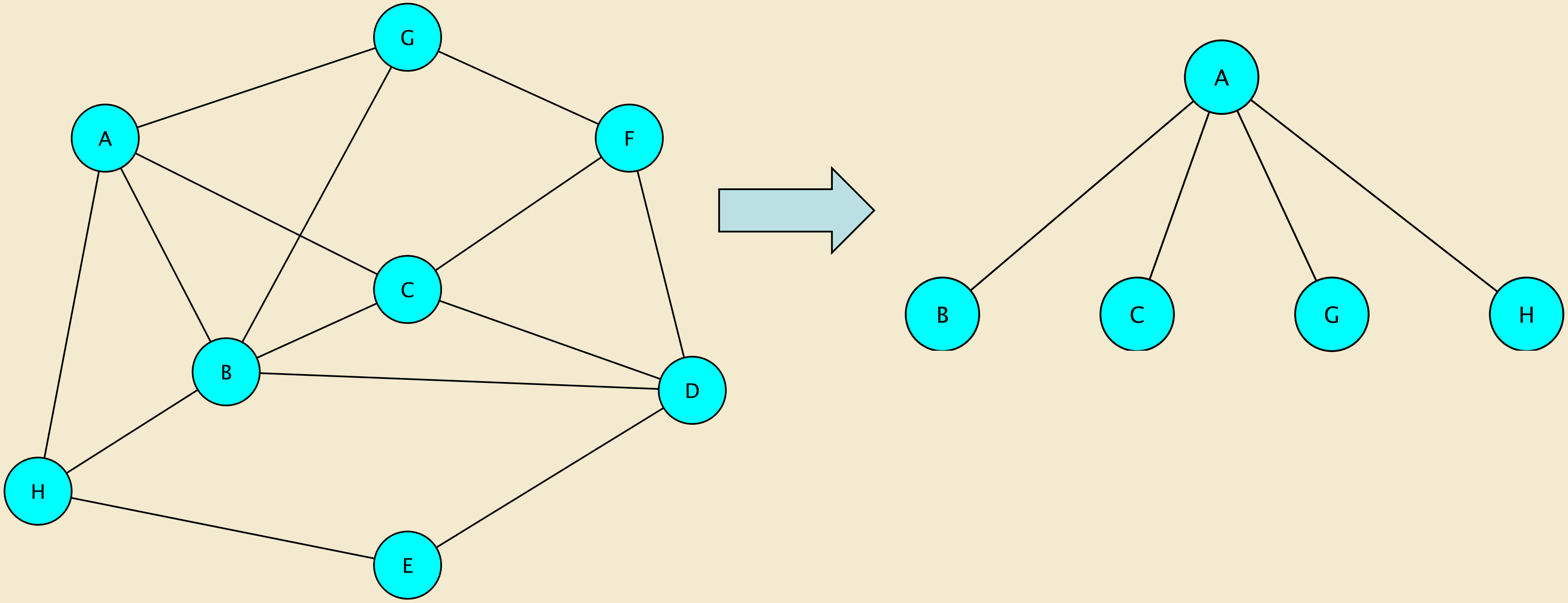
A similar argument shows that if u is visited by BFS(G,v) then u is reachable from v

BFS Reflections

- The BFS algorithm traced out a tree T_v: the edges connecting a visited vertex to (as yet) unvisited neighbors
- T_v is called a *BFS tree of G with root v* (or *from v*)
- The vertices of T_v are visited in *level-order*
- Every path in T_v from v to a vertex u is a *shortest possible path* from v to u
    - That is, the path has length d(v,u)

BFS Reflections : Example

Assuming neighbors are visited alphabetically



Summary and Observations

- An undirected graph models a symmetric relationship between *entities* (vertices)
- Local features of the graph (e.g. : neighbors) can be used to determine global features of the graph (e.g. : distance, connectedness, …)
- Graph algorithms often explore the graph by following sequences of edges (paths)
- An enormous range of problems can be modeled as graph problems