# CSCI 136
# Data Structures &
# Advanced Programming

Lecture 21

Fall 2019

Instructor: B&S

# Last Time

- Traversing Binary Trees
- Lab 7: Two Towers

# Today

- Array Representations of (Binary) Trees
- Application: Huffman Encoding
- Priority Queues & Heaps
  - A "somewhat-ordered" data structure
    - Conceptual structure
    - Efficient implementations

# Array-Based Binary Trees

- Encode structure of tree in array indexes
  - Put root at index 0
- Where are children of node i?
  - Children of node i are at 2i+1 and 2i+2
  - Look at example
- Where is parent of node j?
  - Parent of node j is at (j-1)/2

# ArrayTree Tradeoffs

- Why are ArrayTrees good?
  - Save space for links
  - No need for additional memory allocated/garbage collected
  - Works well for full or complete trees
    - Complete: All levels except last are full and all gaps are at right
    - “A *complete* binary tree of height h is a full binary tree with 0 or more of the rightmost leaves of level h removed”
- Why bad?
  - Could waste a lot of space
  - Tree of height of n requires $2^{n+1}-1$ array slots even if only O(n) elements

# Application: Huffman Codes (a CS 256 Preview)

- Computers encode a text as a sequence of bits

## ASCII TABLE

| Decimal | Hex | Char | Decimal | Hex | Char | Decimal | Hex | Char | Decimal | Hex | Char |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | [NULL] | 32 | 20 | [SPACE] | 64 | 40 | @ | 96 | 60 | ` |
| 1 | 1 | [START OF HEADING] | 33 | 21 | ! | 65 | 41 | A | 97 | 61 | a |
| 2 | 2 | [START OF TEXT] | 34 | 22 | " | 66 | 42 | B | 98 | 62 | b |
| 3 | 3 | [END OF TEXT] | 35 | 23 | # | 67 | 43 | C | 99 | 63 | c |
| 4 | 4 | [END OF TRANSMISSION] | 36 | 24 | $ | 68 | 44 | D | 100 | 64 | d |
| 5 | 5 | [ENQUIRY] | 37 | 25 | % | 69 | 45 | E | 101 | 65 | e |
| 6 | 6 | [ACKNOWLEDGE] | 38 | 26 | & | 70 | 46 | F | 102 | 66 | f |
| 7 | 7 | [BELL] | 39 | 27 | ' | 71 | 47 | G | 103 | 67 | g |
| 8 | 8 | [BACKSPACE] | 40 | 28 | ( | 72 | 48 | H | 104 | 68 | h |
| 9 | 9 | [HORIZONTAL TAB] | 41 | 29 | ) | 73 | 49 | I | 105 | 69 | i |
| 10 | A | [LINE FEED] | 42 | 2A | * | 74 | 4A | J | 106 | 6A | j |
| 11 | B | [VERTICAL TAB] | 43 | 2B | + | 75 | 4B | K | 107 | 6B | k |
| 12 | C | [FORM FEED] | 44 | 2C | , | 76 | 4C | L | 108 | 6C | l |
| 13 | D | [CARRIAGE RETURN] | 45 | 2D | - | 77 | 4D | M | 109 | 6D | m |
| 14 | E | [SHIFT OUT] | 46 | 2E | . | 78 | 4E | N | 110 | 6E | n |
| 15 | F | [SHIFT IN] | 47 | 2F | / | 79 | 4F | O | 111 | 6F | o |
| 16 | 10 | [DATA LINK ESCAPE] | 48 | 30 | 0 | 80 | 50 | P | 112 | 70 | p |
| 17 | 11 | [DEVICE CONTROL 1] | 49 | 31 | 1 | 81 | 51 | Q | 113 | 71 | q |
| 18 | 12 | [DEVICE CONTROL 2] | 50 | 32 | 2 | 82 | 52 | R | 114 | 72 | r |
| 19 | 13 | [DEVICE CONTROL 3] | 51 | 33 | 3 | 83 | 53 | S | 115 | 73 | s |
| 20 | 14 | [DEVICE CONTROL 4] | 52 | 34 | 4 | 84 | 54 | T | 116 | 74 | t |
| 21 | 15 | [NEGATIVE ACKNOWLEDGE] | 53 | 35 | 5 | 85 | 55 | U | 117 | 75 | u |
| 22 | 16 | [SYNCHRONOUS IDLE] | 54 | 36 | 6 | 86 | 56 | V | 118 | 76 | v |
| 23 | 17 | [ENG OF TRANS. BLOCK] | 55 | 37 | 7 | 87 | 57 | W | 119 | 77 | w |
| 24 | 18 | [CANCEL] | 56 | 38 | 8 | 88 | 58 | X | 120 | 78 | x |
| 25 | 19 | [END OF MEDIUM] | 57 | 39 | 9 | 89 | 59 | Y | 121 | 79 | y |
| 26 | 1A | [SUBSTITUTE] | 58 | 3A | : | 90 | 5A | Z | 122 | 7A | z |
| 27 | 1B | [ESCAPE] | 59 | 3B | ; | 91 | 5B | [ | 123 | 7B | { |
| 28 | 1C | [FILE SEPARATOR] | 60 | 3C | < | 92 | 5C | \ | 124 | 7C | \| |
| 29 | 1D | [GROUP SEPARATOR] | 61 | 3D | = | 93 | 5D | ] | 125 | 7D | } |
| 30 | 1E | [RECORD SEPARATOR] | 62 | 3E | > | 94 | 5E | ^ | 126 | 7E | ~ |
| 31 | 1F | [UNIT SEPARATOR] | 63 | 3F | ? | 95 | 5F | _ | 127 | 7F | [DEL] |

# Huffman Codes

- Goal: Encode a text as a sequence of bits
- Normally, use ASCII: 1 character = 8 bits (1 byte)
  - Allows for $2^8 = 256$ different characters
- 'A' = 01000001, 'B' = 01000010
- Space to store "AN_ANTARCTIC_PENGUIN"
  - 20 characters -> 20*8 bits = 160 bits
- Is there a better way?
  - Only 11 symbols are used (ANTRCIPEGU_)
  - Only need 4 bits per symbol (since $2^4 > 11$)!
    - 20*4 = 80 bits instead of 160!
  - Can we still do better??

# Huffman Codes

- Example
  - AN_ANTARCTIC_PENGUIN
  - Compute letter frequencies

| A | C | E | G | I | N | P | R | T | U | _ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 2 | 4 | 1 | 1 | 2 | 1 | 2 |

- Key Idea: Use fewer bits for most common letters

| A | C | E | G | I | N | P | R | T | U | _ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 2 | 4 | 1 | 1 | 2 | 1 | 2 |
| 110 | 111 | 1011 | 1000 | 000 | 001 | 1001 | 1010 | 0101 | 0100 | 011 |

- Uses 67 bits to encode entire string

# Huffman Codes

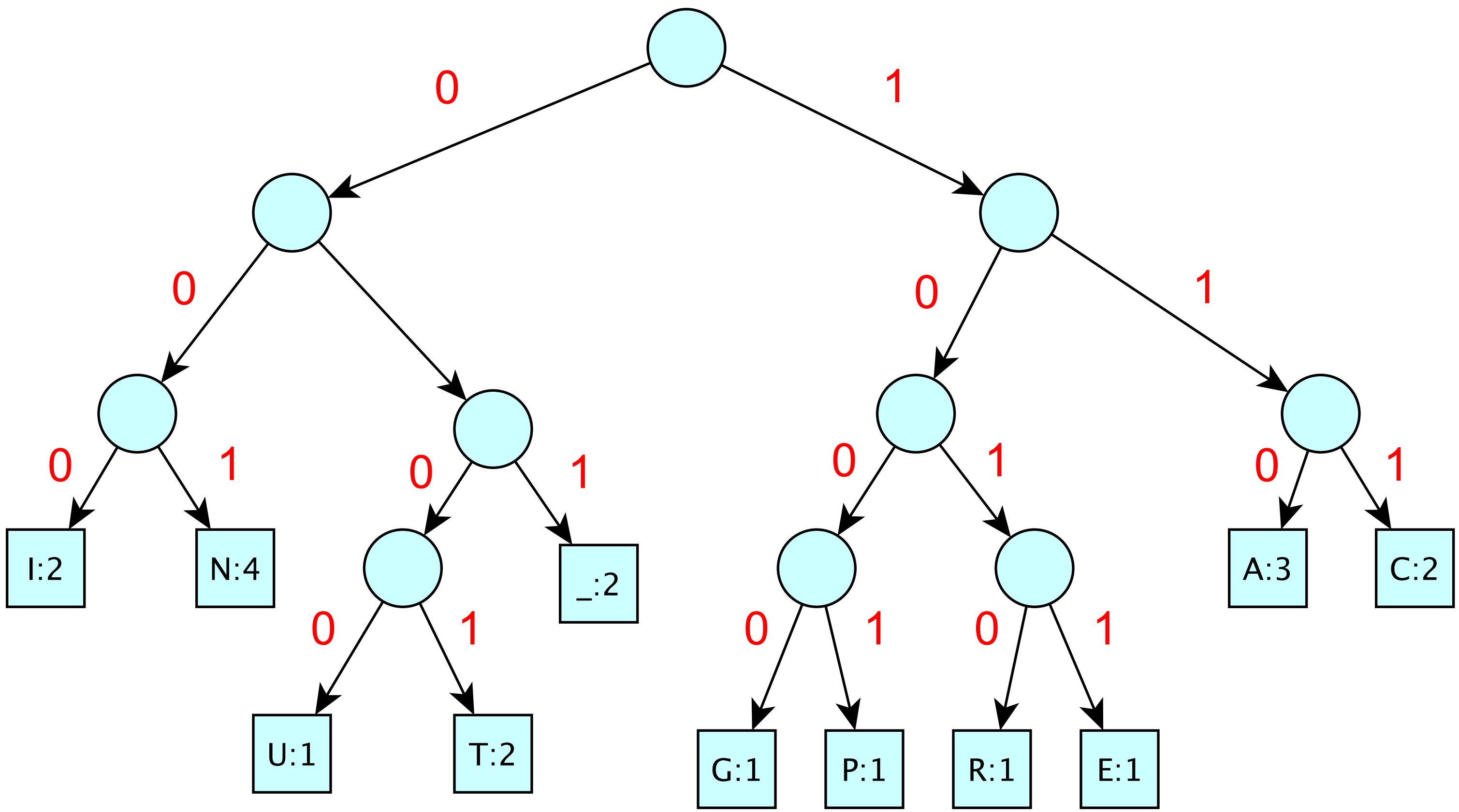
| A | C | E | G | I | N | P | R | T | U | _ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 2 | 4 | 1 | 1 | 2 | 1 | 2 |
| 110 | 111 | 1011 | 1000 | 000 | 001 | 1001 | 1010 | 0101 | 0100 | 011 |

- Uses 67 bits to encode entire string
- Can we do better?

| A | C | E | G | I | N | P | R | T | U | _ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 2 | 4 | 1 | 1 | 2 | 1 | 2 |
| 100 | 010 | 1100 | 1101 | 011 | 101 | 0001 | 0000 | 001 | 1110 | 1111 |

- Uses 67 bits to encode entire string

# The Encoding Tree




Left = 0; Right = 1

# Features of Good Encoding

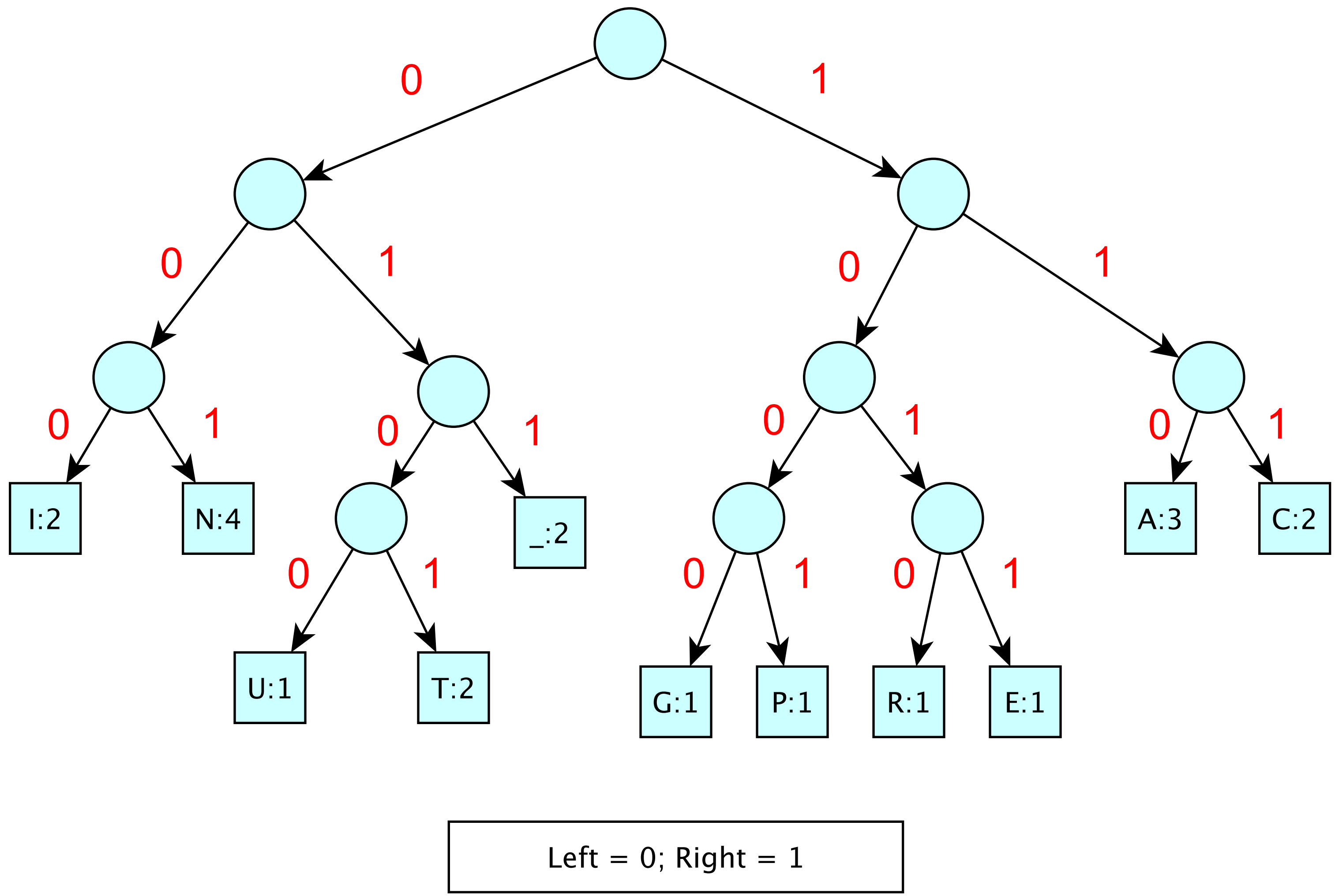
- Prefix property: No encoding is a prefix of another encoding (letters appear at leaves)
- No internal node has a single child
- Nodes with lower frequency have greater depth

- All optimal length unambiguous encodings have these features

# Huffman Encoding

- Input: symbols of alphabet with frequencies
- Huffman encode as follows
  - Create a single-node tree for each symbol: key is frequency; value is letter
  - while there is more than one tree
    - Find two trees T1 and T2 with lowest keys
    - Merge them into new tree T with dummy value and key= T1.key+ T2.key
- Theorem: The tree computed by Huffman is an optimal encoding for given frequencies

# The Encoding Tree



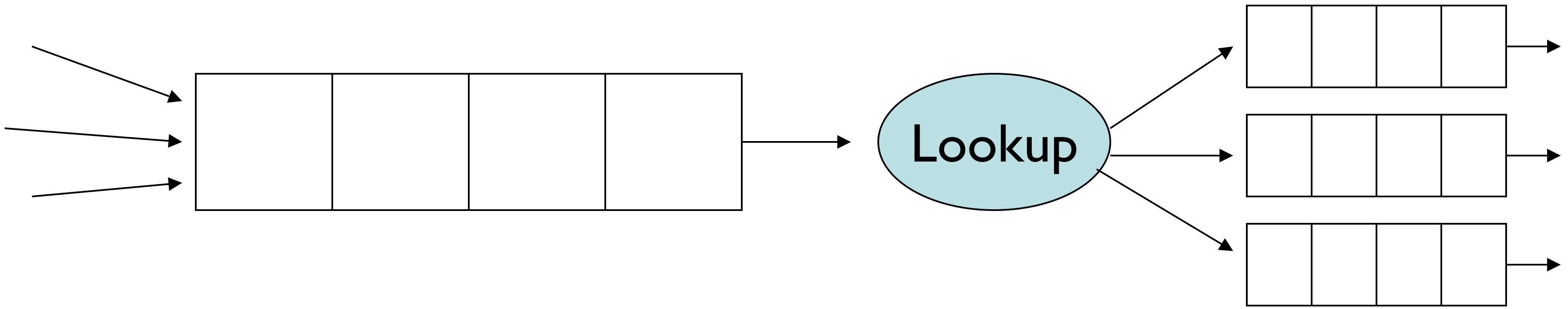
# How To Implement Huffman

- Keep a Vector of Binary Trees
- Sort them by decreasing frequency
  - Removing two smallest frequency trees is fast
- Insert merged tree into correct sorted location in Vector
- Running Time:
  - $O(n \log n)$ for initial sorting
  - $O(n^2)$ for rest: $O(n)$ re-insertions of merged trees
- Can we do better…?

# What Huffman Encoder Needs

- A structure S to hold items with *priorities*
- S should support operations
  - add(E item);   // add an item
  - E removeMin();  // remove min priority item
- S should be designed to make these two operations fast
- If, say, they both ran in O(log n) time, the Huffman algorithm would take O(n log n) time instead of O($n^2$)!
- We've seen this situation before….

# Priority Queues




## Packet Sources May Be Ordered by Sender

| | |
|---|---|
| `sysnet.cs.williams.edu` | `priority = 1 (best)` |
| `bull.cs.williams.edu` | `2` |
| `yahoo.com` | `10` |
| `spammer.com` | `100 (worst)` |

# Priority Queues

- Priority queues are also used for:
  - Scheduling processes in an operating system
    - Priority is function of time lost + process priority
  - Order services on server
    - Backup is low priority, so don't do when high priority tasks need to happen
  - Scheduling future events in a simulation
  - Medical waiting room
  - Huffman codes - order by tree size/weight
  - A variety of graph/network algorithms
  - To roughly order choices that are generated out of order

# Priority Queues

- Name is misleading: They are **not FIFO**
- Always dequeue object with **highest priority** (smallest rank) regardless of when it was enqueued
- Data can be received/inserted in any order, but it is always returned/removed according to priority
- Like ordered structures (i.e., OrderedVectors and OrderedLists), PQs require comparisons of values

# An Apology

- On behalf of computer scientists everywhere, I'd like to apologize for the confusion that inevitably results from the fact that

  Higher Priority     Lower Rank

- The PQ removes the *lowest ranked* value in an ordering: that is, the *highest priority* value!

We're sorry!

# PQ Interface

```
public interface PriorityQueue<E extends Comparable<E>> {
  public E getFirst(); // peeks at minimum element
  public E remove();   // removes minimum element
  public void add(E value); // adds an element
  public boolean isEmpty();
  public int size();
  public void clear();
}
```

# Notes on PQ Interface

- Unlike previous structures, we do not extend any other interfaces
  - Many reasons: For example, it's not clear that there's an obvious iteration order
- PriorityQueue uses Comparables: methods *consume* Comparable parameters and *return* Comparable values
  - Could be made to use Comparators instead…

# Implementing PQs

- Queue?
  - Wouldn't work so well because we can't insert and remove in the "right" way (i.e., keeping things ordered)
- OrderedVector?
  - Keep ordered vector of objects
  - O(n) to add/remove from vector
  - Details in book…
  - Can we do better than O(n)?
- Heap!
  - Partially ordered binary tree

# Heap

- A heap is a special type of tree
- A heap is a tree where:
  - Root holds smallest (highest priority) value
  - Subtrees are also heaps (this is important!)
- So values increase in priority (decrease in value) from leaves to root (from descendant to ancestor)
- *Invariant for nodes*
  - node.value() >= node.parent.value()
    - Tree need not be binary….
- Several valid heaps for same data set (no unique representation)

# Inserting into a PQ

- Add new value as a leaf
- "Percolate" it up the tree
  - while (value < parent's value) swap with parent
- This operation preserves the heap property since new value was the only one violating heap property
- Efficiency depends upon speed of
  - Finding a place to add new node
  - Finding parent
  - Tree height

# Removing From a PQ

- Find a leaf, delete it, put its *data* in the root
- "Push" *data* down through the tree
  - while ( *data.value* > value of (at least) one child )
    - Swap *data* with data of **smallest** child
- This operation preserves the heap property
- Efficiency depends upon speed of
  - Finding a leaf
  - Finding locations of children
  - Determining child with smallest value
  - Height of tree

# Implementing Heaps

- VectorHeap
  - Use conceptual array representation of BT (ArrayTree)
  - But use extensible Vector instead of array (makes adding elements easier)
  - Note:
    - Root of tree is location 0 of Vector
    - Children of node in location i are in locations 2i+1 (left) and 2i+2 (right)
    - Parent of node i is in location (i-1)/2

# Implementing Heaps

- Features
  - No gaps in array (array is *complete*)-- why?
    - We always add in next available array slot (left-most available spot in binary tree;
    - We always remove using “final” leaf
  - *Heap Invariant becomes*
    - data[i] <= data[2i+1]; data[i]<=data[2i+2] (or kids might be null)
  - When elements are added and removed, do small amount of work to “re-heapify”
    - How small? Note: finding a node’s child or parent takes constant time, as does finding “final” leaf or next slot for adding
    - Since this heap corresponds to a full binary tree, the depth of the tree is O(log n), so percolate/pushDown takes O(log n) time!

# VectorHeap Summary

- Let's look at VectorHeap code....

- Add/remove are both O(log n)
- Data is not completely sorted
  - "Partial" order is maintained
- Note: VectorHeap(Vector<E> v)
  - Takes an unordered Vector and uses it to construct a heap
  - How?

# Heapifying A Vector (or array)

- Method I: Top-Down
  - Assume V[0...k] satisfies the heap property
  - Now call percolate on item in location k+1
  - Then V[0..k+1] satisfies the heap property
- Method II: Bottom-up
  - Assume V[k..n] satisfies the heap property
  - Now call pushDown on item in location k-1
  - Then V[k-1..n] satisfies heap property

# Top-Down vs Bottom-Up

- Top-down heapify: elements at depth d may be swapped d times: Total # of swaps is at most

$$\sum_{d=0}^{h} d2^d = (h-1)2^{h+1} + 2 = (\log n - 1)2n + 2$$

- This is O(n log n)
- Some intuition: most of the elements are in the lowest levels of the tree, so each of them might have to move to root: O(log n) swaps per element

# Top-Down vs Bottom-Up

- Bottom-up heapify: elements at depth d may be swapped h-d times: Total # of swaps is at most

$$\sum_{d=0}^{h}(h-d)2^d = 2^{h+1} - h - 2 = 2n - \log n + 2$$

  - This is O(n) --- beats top-down!
  - Some intuition: most of the elements are in the lowest levels of the tree, so each of them will only be pushed down (swapped) a small number of times SO COOL!!!

# Some Sums

$$\sum_{d=0}^{d=k} 2^d = 2^{k+1} - 1$$

$$\sum_{d=0}^{d=k} r^d = (r^{k+1}-1)/(r-1)$$

$$\sum_{d=0}^{d=k} d * 2^d = (k-1) * 2^{k+1} + 2$$

$$\sum_{d=0}^{d=k} (k-d) * 2^d = 2^{k+1} - k - 2$$

All of these can be proven by (weak) induction.

Try these to hone your skills

The second sum is called a geometric series. It works for any r≠0

# HeapSort

- Heaps yield another O(n log n) sort method
- To HeapSort a Vector "in place"
  - Perform bottom-up heapify on the reverse ordering: that is: highest rank/lowest priority elements are near the root (low end of Vector)
  - Now repeatedly remove elements to fill in Vector from tail to head
    - For(int i = v.size() – 1; i > 0; i--)
      - RemoveMin from v[0..i] // v[i] is now not in heap
      - Put removed value in location v[i]

# Skew Heap

- What if heaps are not complete BTs?
- We can implement PQs using skew heaps instead of “regular” complete heaps
- Key differences:
  - Rather than use Vector as underlying data structure, use BT
  - Need a merge operation that merges two heaps together into one heap
- Details in book