CSCI 136
Data Structures & Advanced Programming

Lecture 19

Fall 2019

Instructor: Bill & Sam

Administration

- CS prereg info session today 2:35 Wege (TCL 123)

Last Time

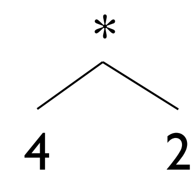
- Trees
  - Vocabulary ☹
  - Expression Trees
    - Recursive evaluation
  - Implementation

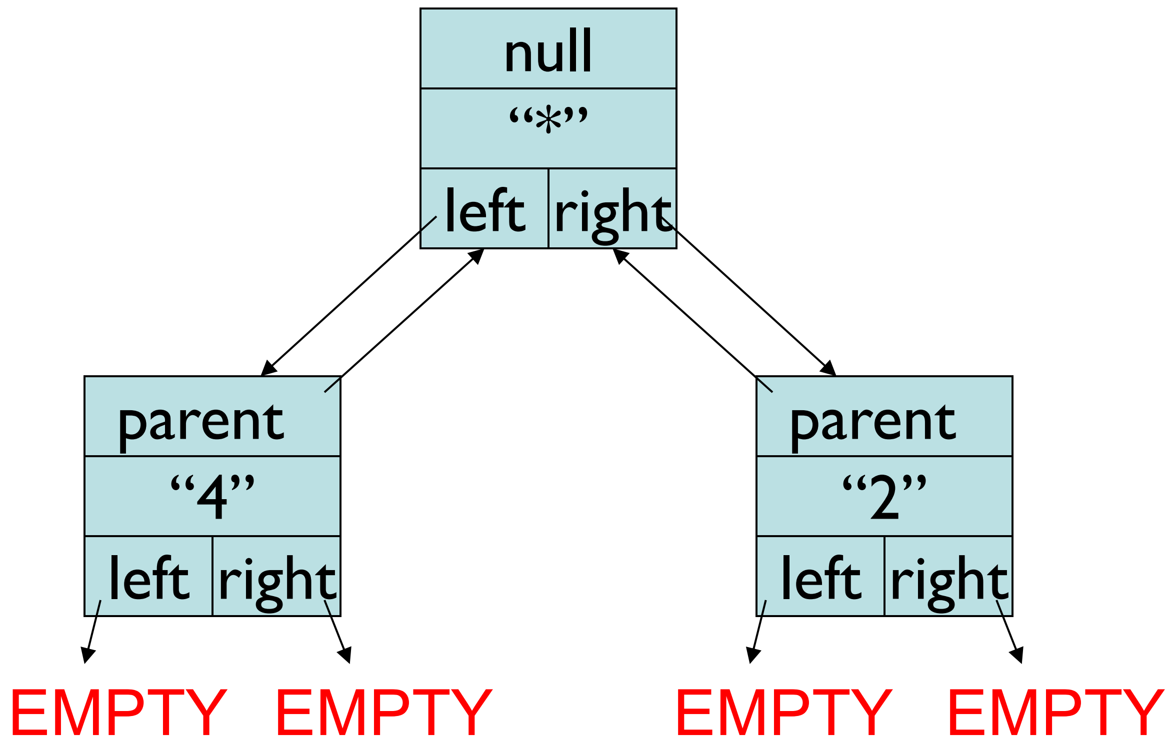
Today

- Implementation (finish)
- Recursion/induction
- Applications: Decision Trees
- Trees with more than 2 children
  - Representations
- Traversing Binary Trees
  - As methods taking a BinaryTree parameter
  - With Iterators

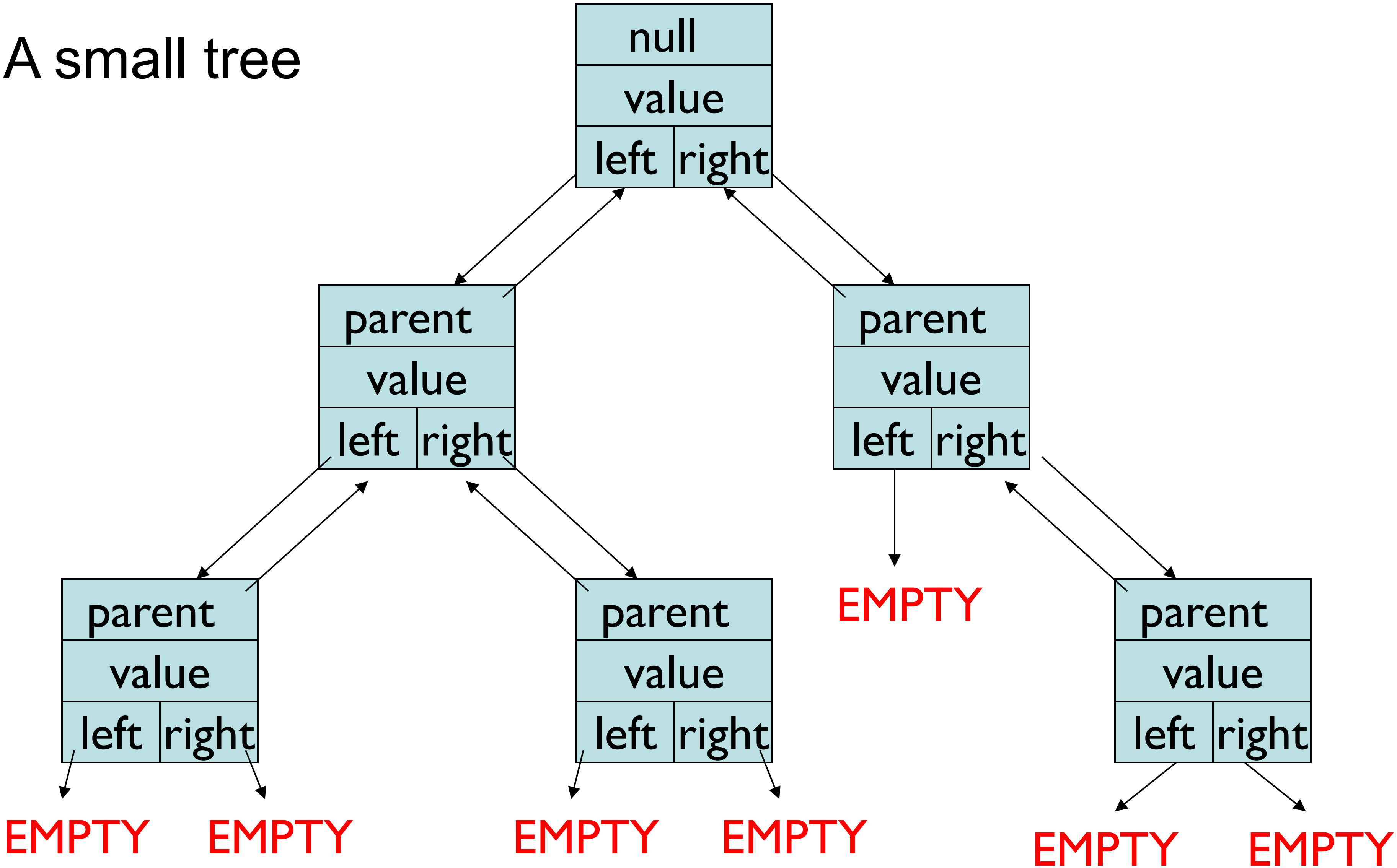
Implementing BinaryTree

- BinaryTree class
  - Instance variables
    - BT parent, BT left, BT right, E value





A small tree



EMPTY != null!

Implementing BinaryTree

- Many (!) methods: See BinaryTree javadoc page
- All “left” methods have equivalent “right” methods
  - public BinaryTree()
    - // generates an empty node (EMPTY)
    - // parent and value are null, left=right=this
  - public BinaryTree(E value)
    - // generates a tree with a non-null value and two empty (EMPTY) subtrees
  - public BinaryTree(E value, BinaryTree<E> left, BinaryTree<E> right)
    - // returns a tree with a non-null value and two subtrees
  - public void setLeft(BinaryTree<E> newLeft)
    - // sets left subtree to newLeft
    - // re-parents newLeft by calling newLeft.setParent(this)
  - protected void setParent(BinaryTree<E> newParent)
    - // sets parent subtree to newParent
    - // called from setLeft and setRight to keep all “links” consistent

Implementing BinaryTree

- Methods:
  - public BinaryTree<E> left()
    - // returns left subtree
  - public BinaryTree<E> parent()
    - // post: returns reference to parent node, or null
  - public boolean isLeftChild()
    - // returns true if this is a left child of parent
  - public E value()
    - // returns value associated with this node
  - public void setValue(E value)
    - // sets the value associated with this node
  - public int size()
    - // returns number of (non-empty) nodes in tree
  - public int height()
    - // returns height of tree rooted at this node
  - But where's "remove" or "add"?!?!

BT Questions/Proofs

- Prove
  - The number of nodes at depth n is at most 2^n
  - The number of nodes in tree of height n is at most $2^{n+1} - 1$
  - A tree with n nodes has exactly $n - 1$ edges
  - The size() method works correctly
  - The height() method works correctly
  - The isFull() method works correctly

BT Questions/Proofs

Prove: Number of nodes at depth $d \geq 0$ is at most 2^d

Idea: Induction on depth d of nodes of tree

Base case: $d = 0$: 1 node; $1 = 2^0$ ✓

Induction Hyp.: For some $d \geq 0$, there are at most 2^d nodes at depth d

Induction Step: Consider depth d. There are at most 2 nodes at depth $d + 1$ for every node at depth d.

Therefore it has at most $2 * 2^d = 2^{d+1}$ nodes ✓

BT Questions/Proofs

Prove that any tree on $n \geq 1$ nodes has $n-1$ edges

Idea: Induction on number of nodes

Base case: $n = 1$. There are no edges ✓

Induction Hyp: Assume that, for some $n \geq 1$, every tree on n nodes has exactly $n-1$ edges.

Induction Step: Let T have $n+1$ nodes. Show it has exactly n edges.

- Remove a leaf v (and its single edge) from T
- Now T has n nodes, so it has $n-1$ edges
- Now add v (and its single edge) back, giving $n+1$ nodes and n edges.

BT Questions/Proofs

Prove that BinaryTree method size() is correct.

- Let n be the number of nodes in the tree T

Base case: $n = 0$. T is empty---size() returns 0✓

Induction Hyp: Assume size() is correct for *all trees* having *at most* n nodes.

Induction Step: Assume T has $n + 1$ nodes

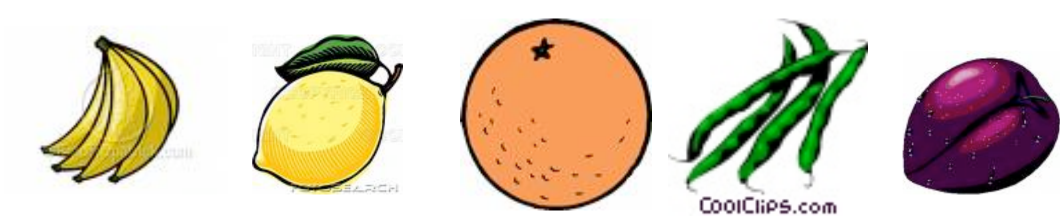
- Then left/right subtrees each have *at most* n nodes
- So size() returns correct value for each subtree
- And the size of T is 1 + size of left subtree + size of right subtree✓

Representing Knowledge

- Trees can be used to represent knowledge
  - Example: InfiniteQuestions game
    - Let's play!
- We often call these trees decision trees
  - Leaf: object
  - Internal node: question to distinguish objects
- Two methods: play() and learn()
  - Play: Move down decision tree until we reach a leaf
    - Check to see if the leaf is correct
  - Learn: If not correct, add question, make new and old objects children
- Let's look at the code

Building Decision Trees

- Gather/obtain data
- Analyze data
  - Make greedy choices: Find good questions that divide data into halves (or as close as possible)
- Construct tree with shortest height
- In general this is a *hard* problem!
- Example

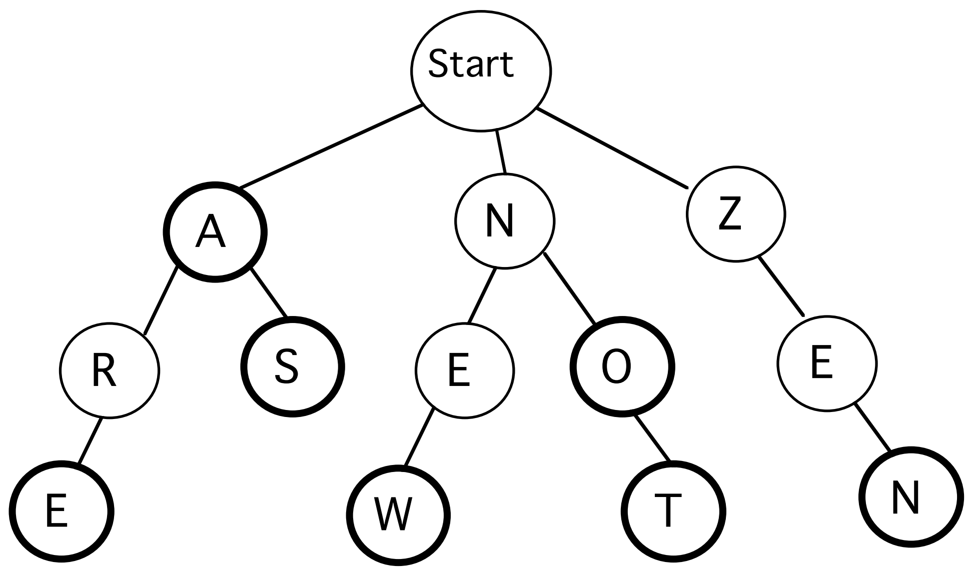


Representing Arbitrary Trees

- What if nodes can have many children?
  - Example: Game trees
- Replace left/right node references with a list of children (Vector, SLL, etc)
  - Allows getting "i^{th}" child
- Should provide method for getting degree of a node
- Degree 0 Empty list No children Leaf

Lab 9 Preview : Lexicon

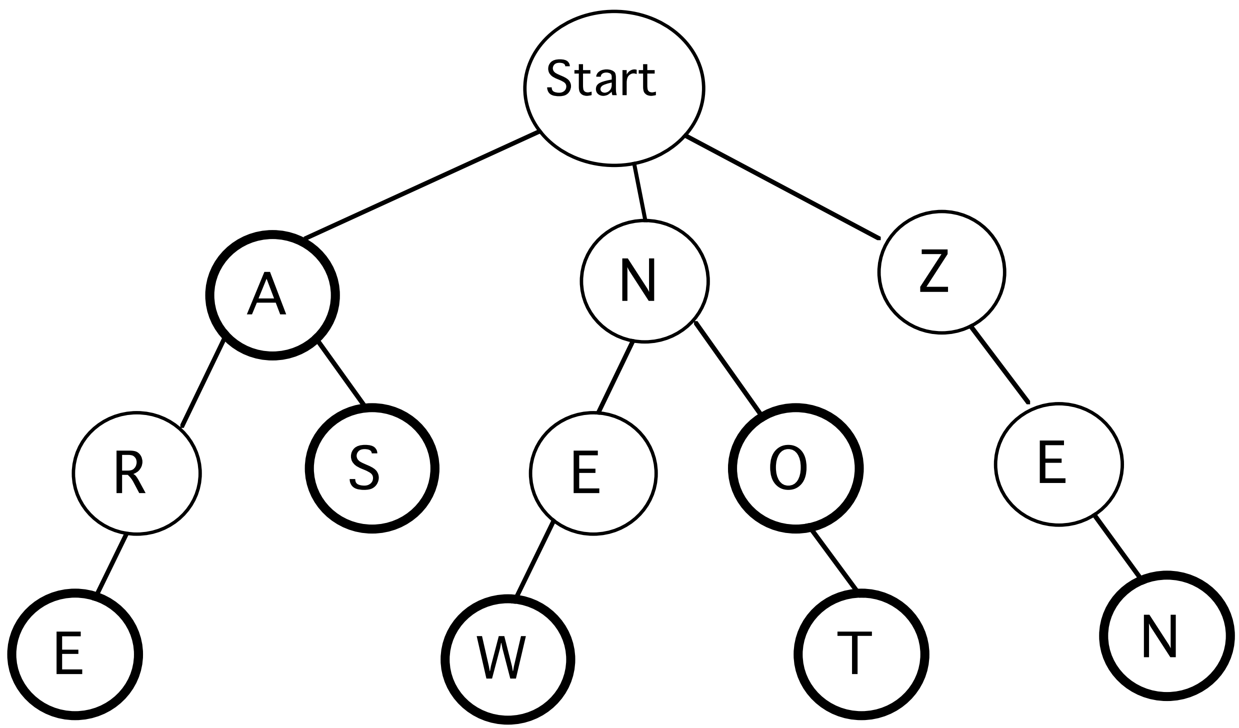
- Goal: Build a data structure that can efficiently store and search a large set of words
- A special kind of tree called a *trie*



Lab 9 Preview : Tries

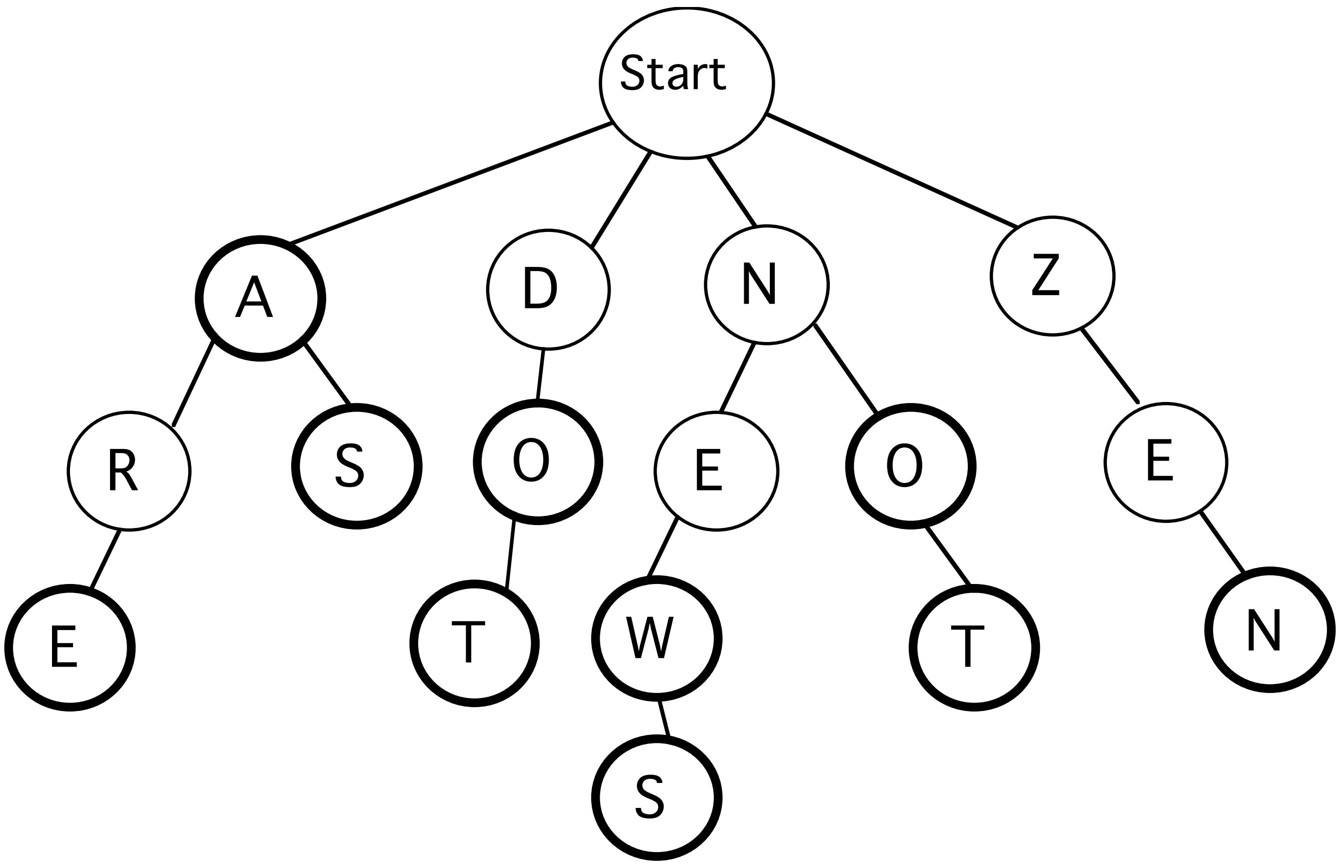
- A trie is a tree that stores words where
  - Each node holds a letter
  - Some nodes are “word” nodes (dark circles)
  - Any path from the root to a word node describes one of the stored words
  - All paths from the root form prefixes of stored words (a word is considered a prefix of itself)

Tries



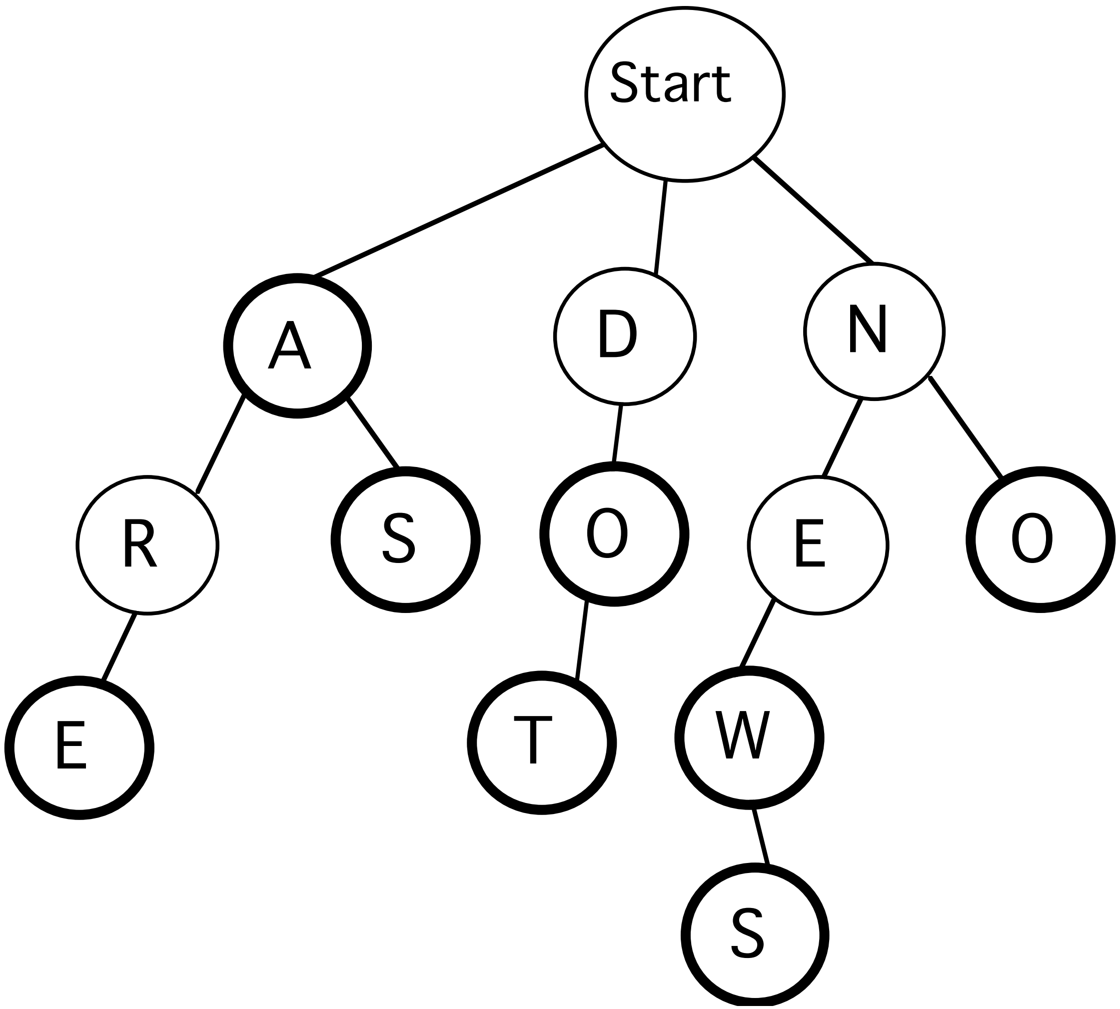
Now add “dot” and “news”

Tries



Now remove “not” and “zen”

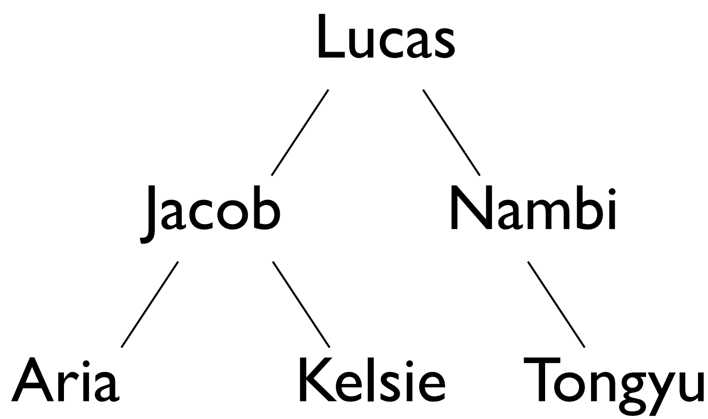
Tries



Tree Traversals

- In linear structures, there are only a few basic ways to traverse the data structure
  - Start at one end and visit each element
  - Start at the other end and visit each element
- How do we traverse binary trees?
  - (At least) four reasonable mechanisms

Tree Traversals



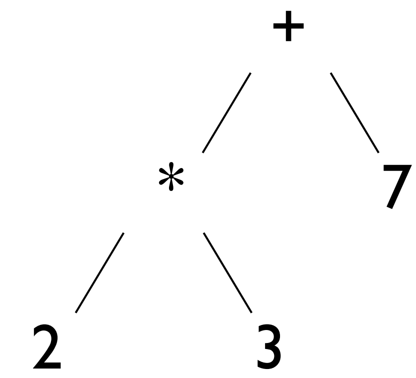
In-order: Aria, Jacob, Kelsie, Lucas, Nambi, Tongyu

Pre-order: Lucas, Jacob, Aria, Kelsie, Nambi, Tongyu

Post-order: Aria, Kelsie, Jacob, Tongyu, Nambi, Lucas,

Level-order: Lucas, Jacob, Nambi, Aria, Kelsie, Tongyu

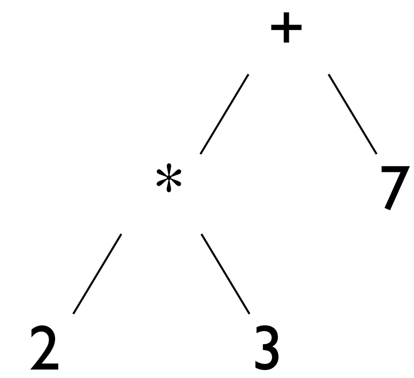
Tree Traversals



- Pre-order
  - Each node is visited before any children. Visit node, then each node in left subtree, then each node in right subtree. (node, left, right)
    - +*237
- In-order
  - Each node is visited after all nodes in left subtree are visited and before any nodes in right subtree. (left, node, right)
    - 2*3+7

(“pseudocode”)

Tree Traversals

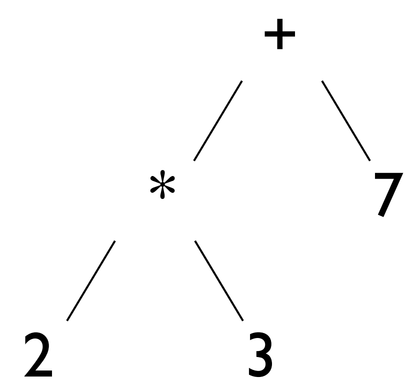


- Post-order
  - Each node is visited after its children are visited. Visit all nodes in left subtree, then all nodes in right subtree, then node itself. (left, right, node)
    - 23*7+
- Level-order (not obviously recursive!)
  - All nodes of level i are visited before nodes of level i+1. (visit nodes left to right on each level)
    - +*723

("pseudocode")

Tree Traversals

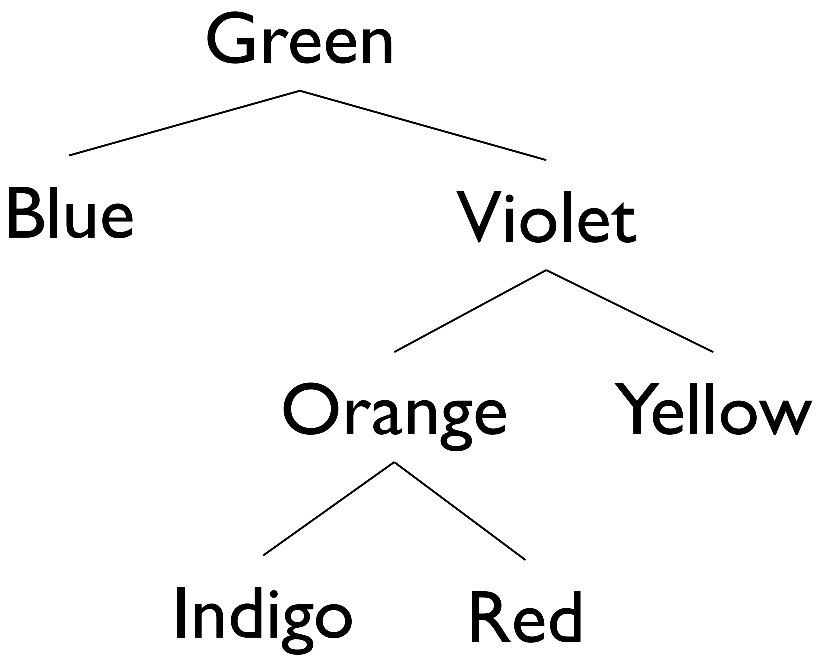
```
public void pre-order(BinaryTree t) {
      if(t.isEmpty()) return;
      touch(t); // some method
      preOrder(t.left());
      preOrder(t.right());
}
```



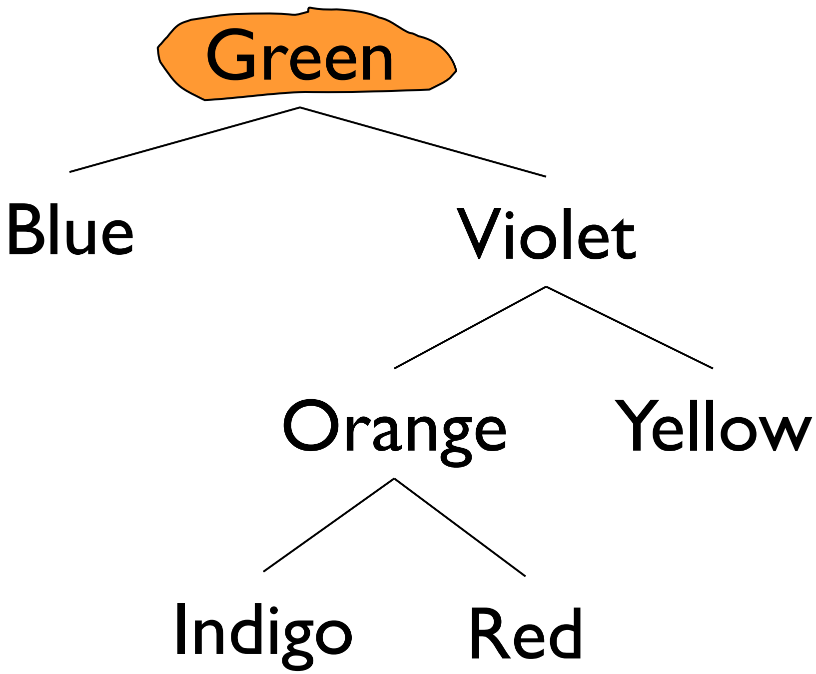
For in-order and post-order: just move touch(t)!

But what about level-order???

Level-Order Traversal

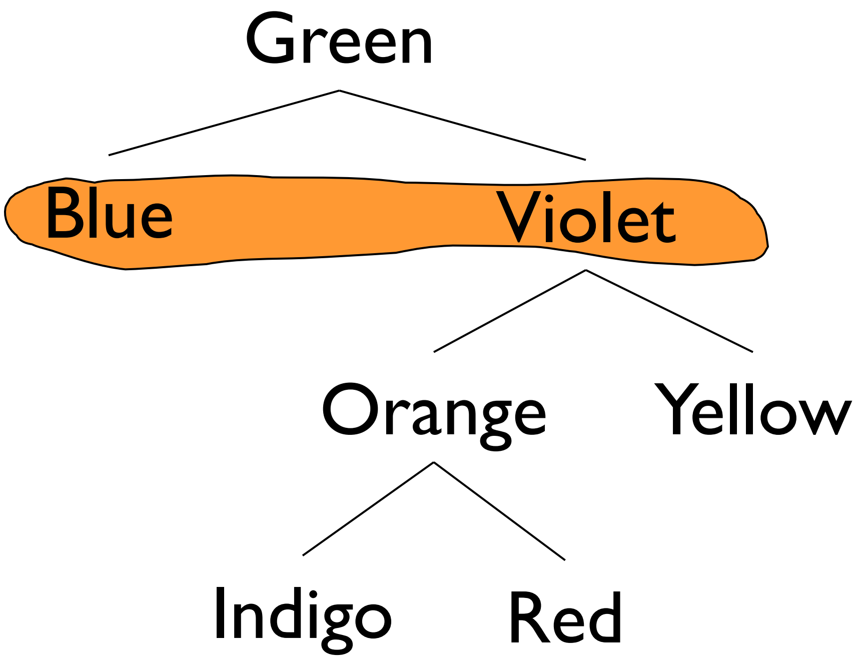


Level-Order Traversal



G

Level-Order Traversal



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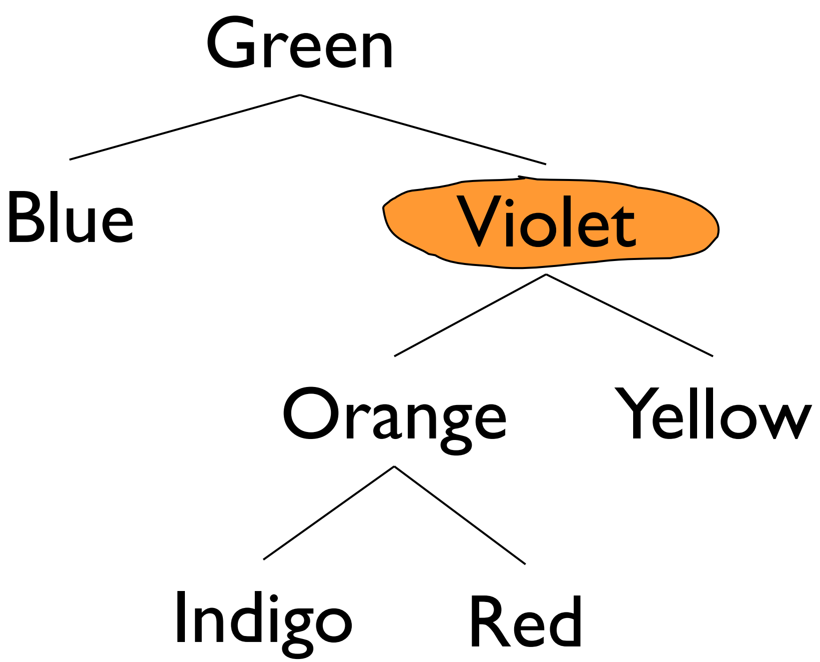
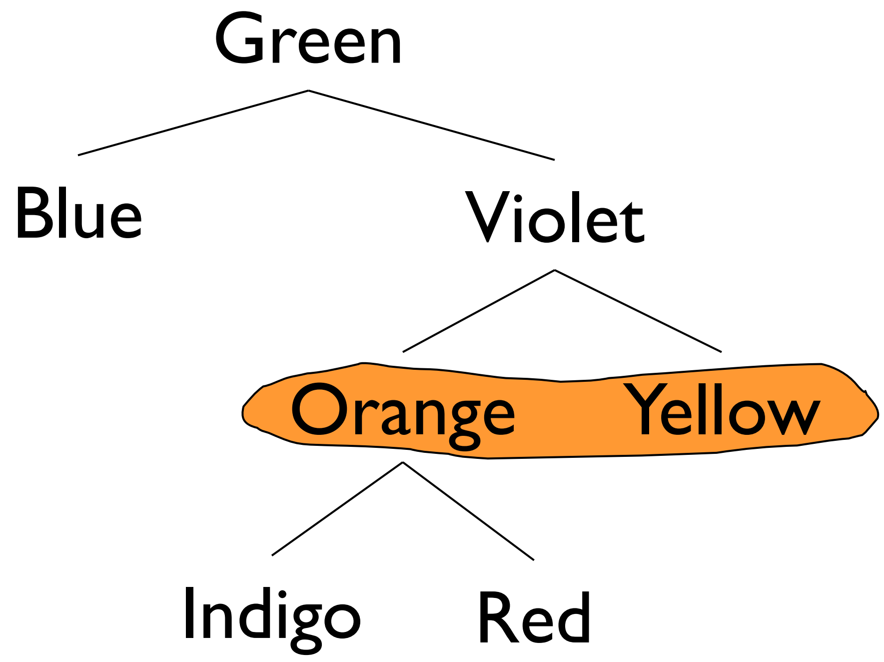
Level-Order Traversal



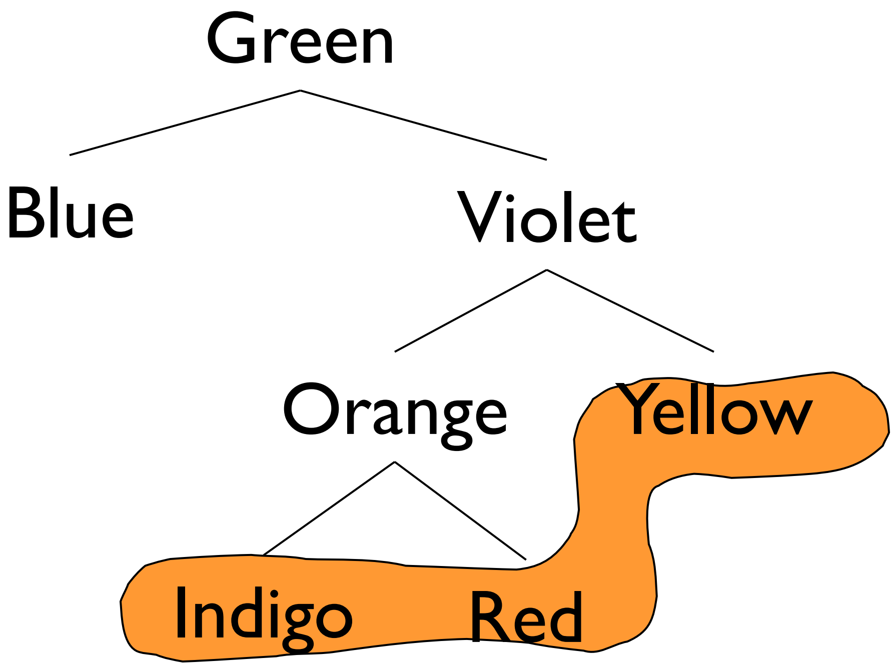
G B

Level-Order Traversal



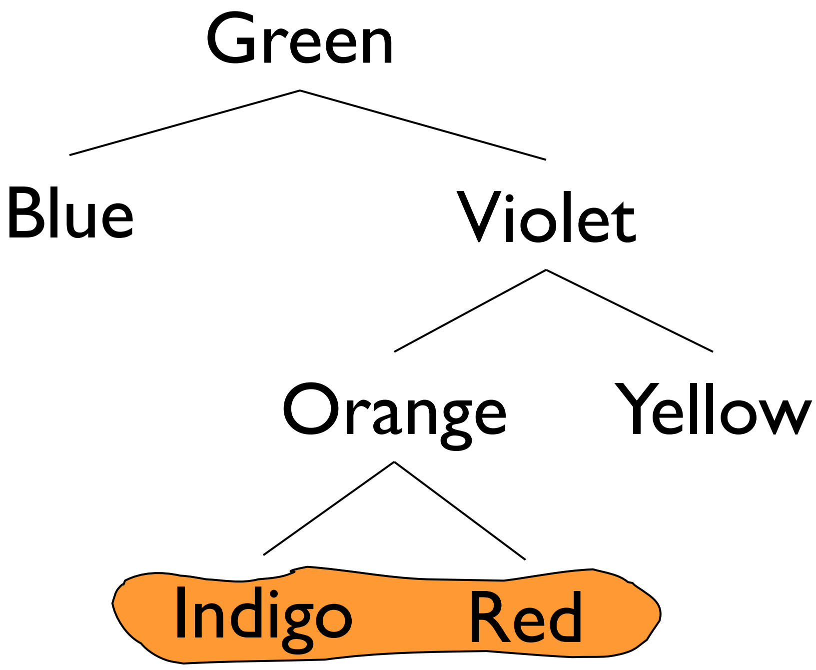
G B V

Level-Order Traversal



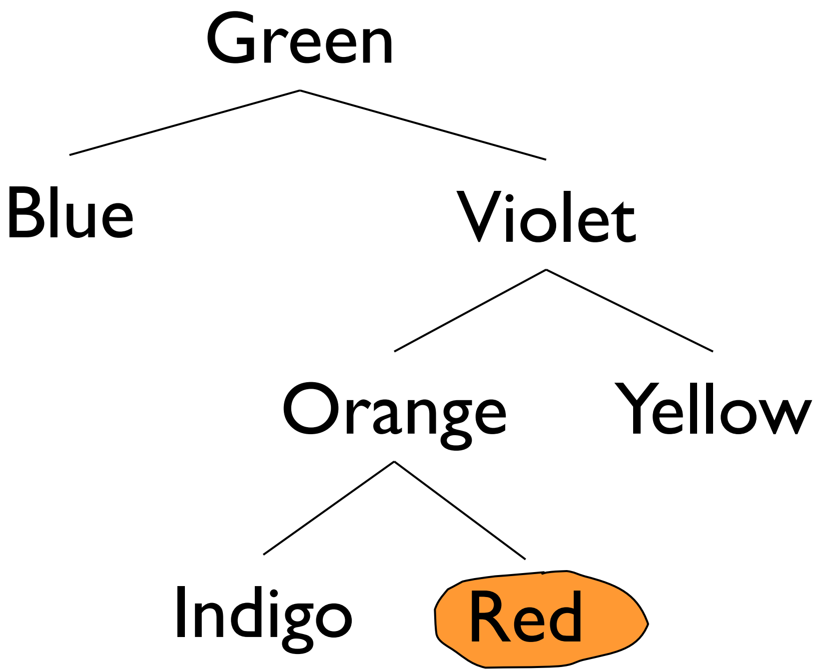
G B V O

Level-Order Traversal



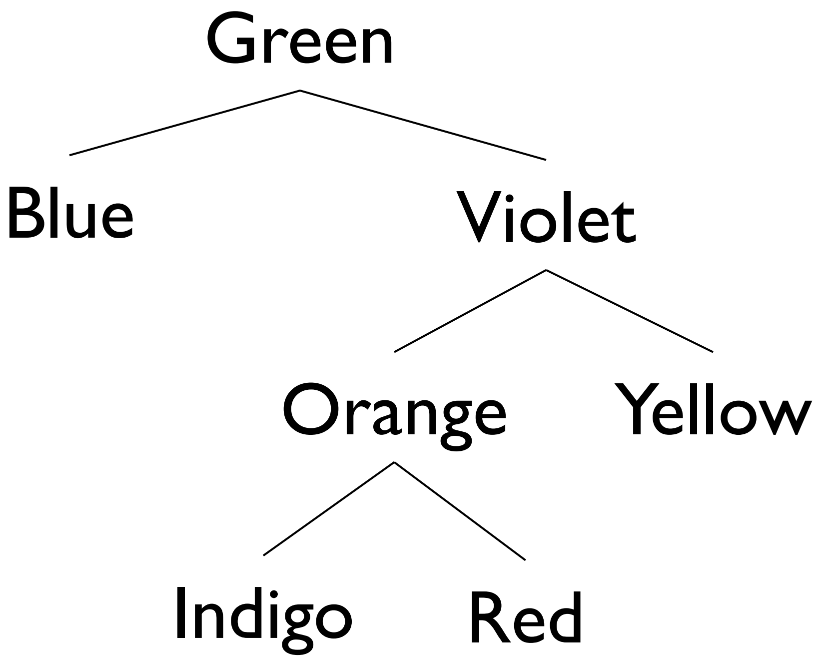
G B V O Y

Level-Order Traversal



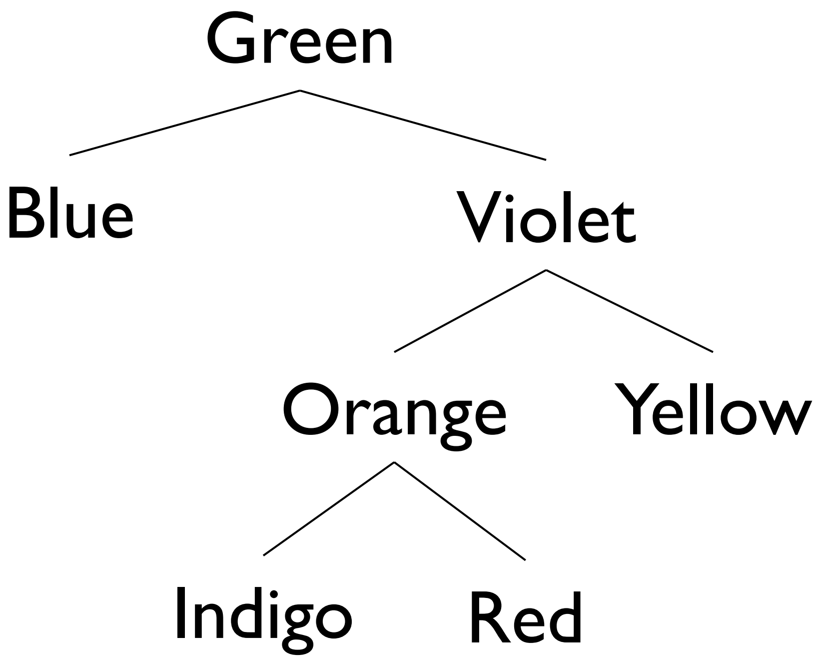
G B V O Y I

Level-Order Traversal

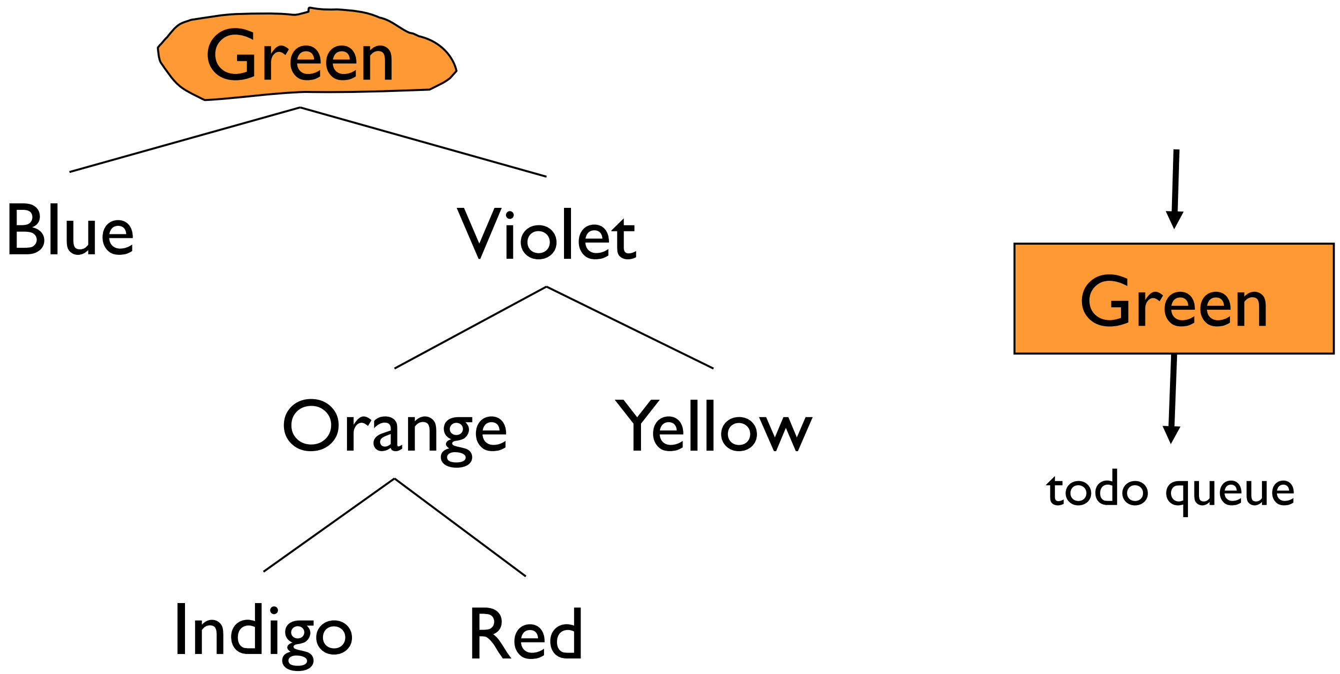


G B V O Y I R

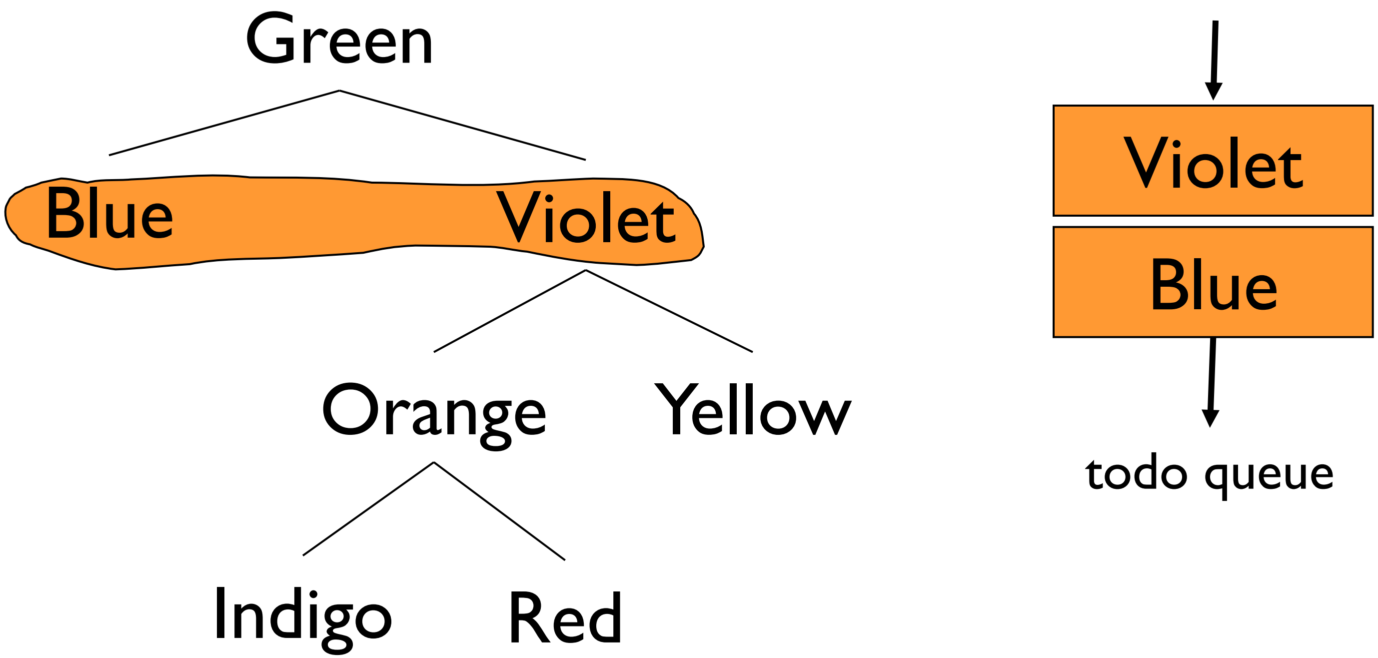
Level-Order Traversal



Level-Order Traversal



Level-Order Traversal



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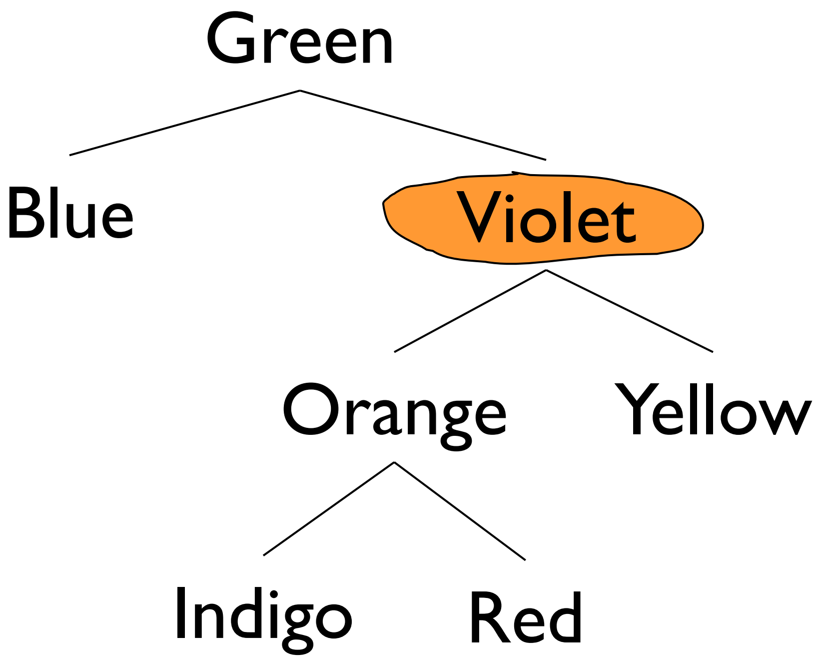
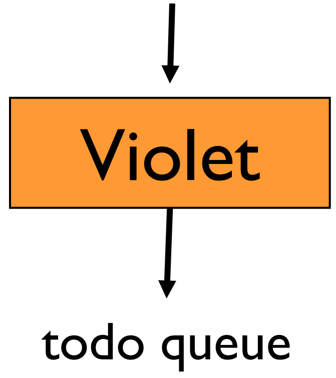
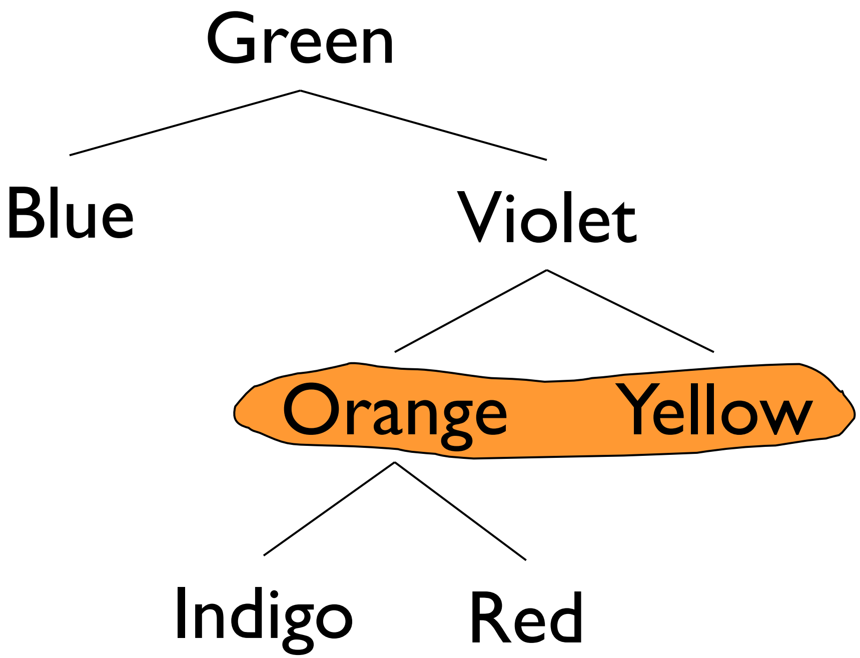
Level-Order Traversal

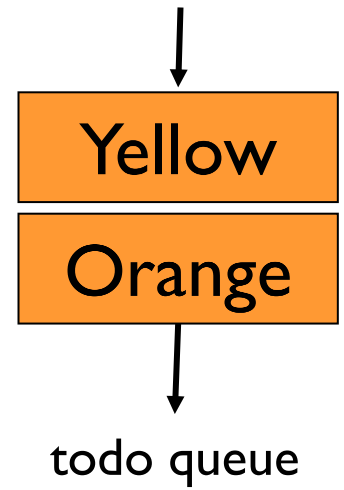




G B

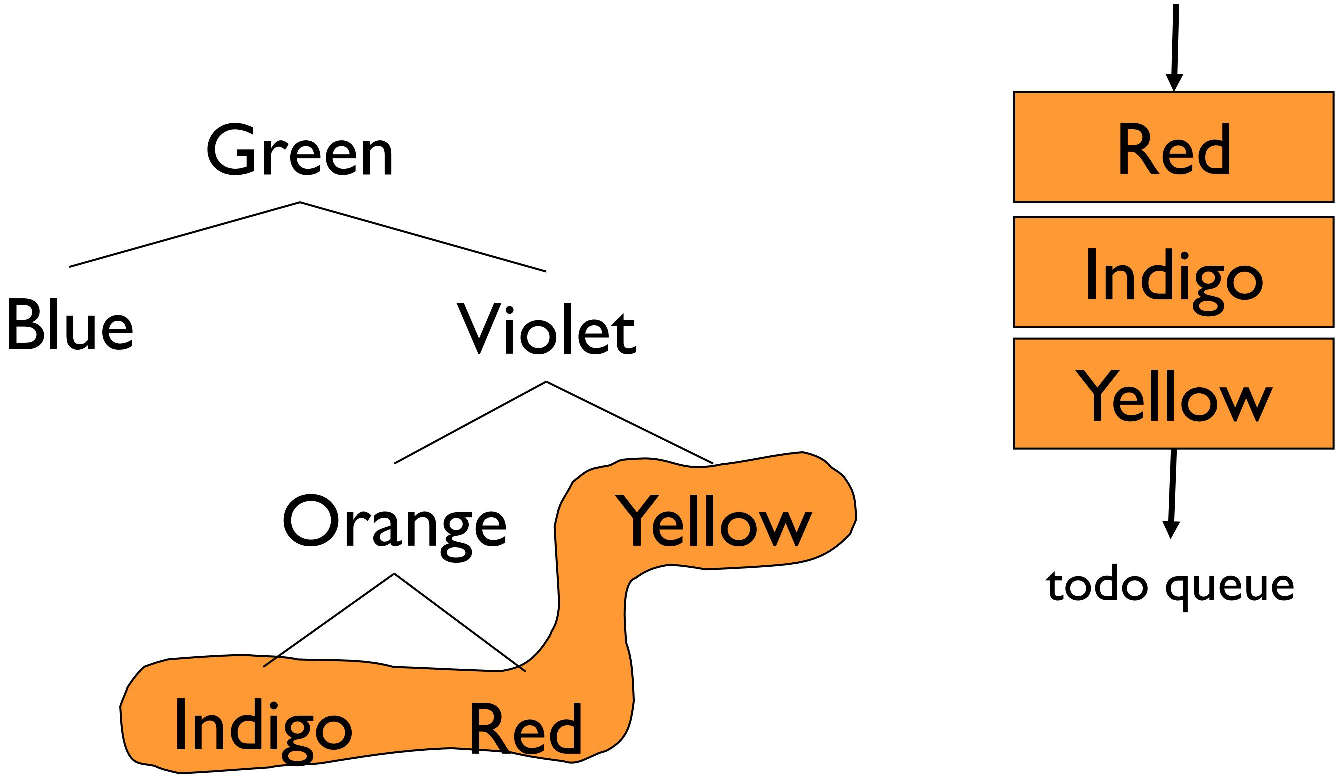
Level-Order Traversal





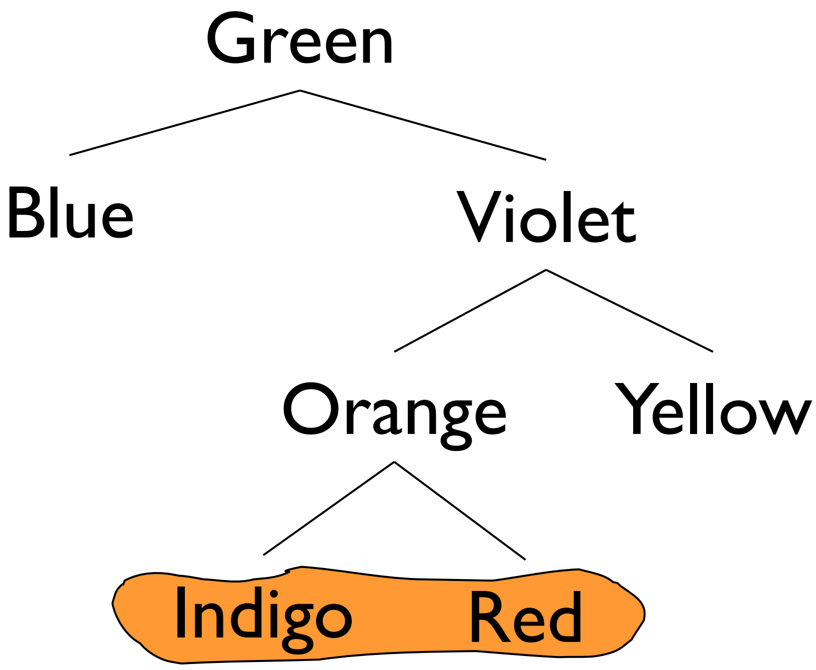
G B V

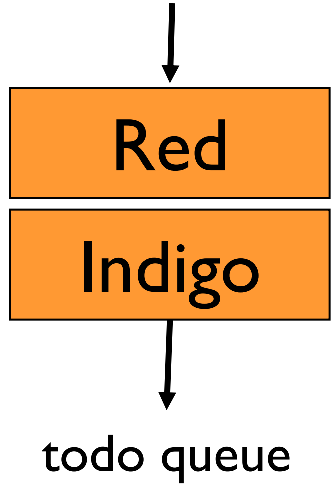
Level-Order Traversal



G B V O

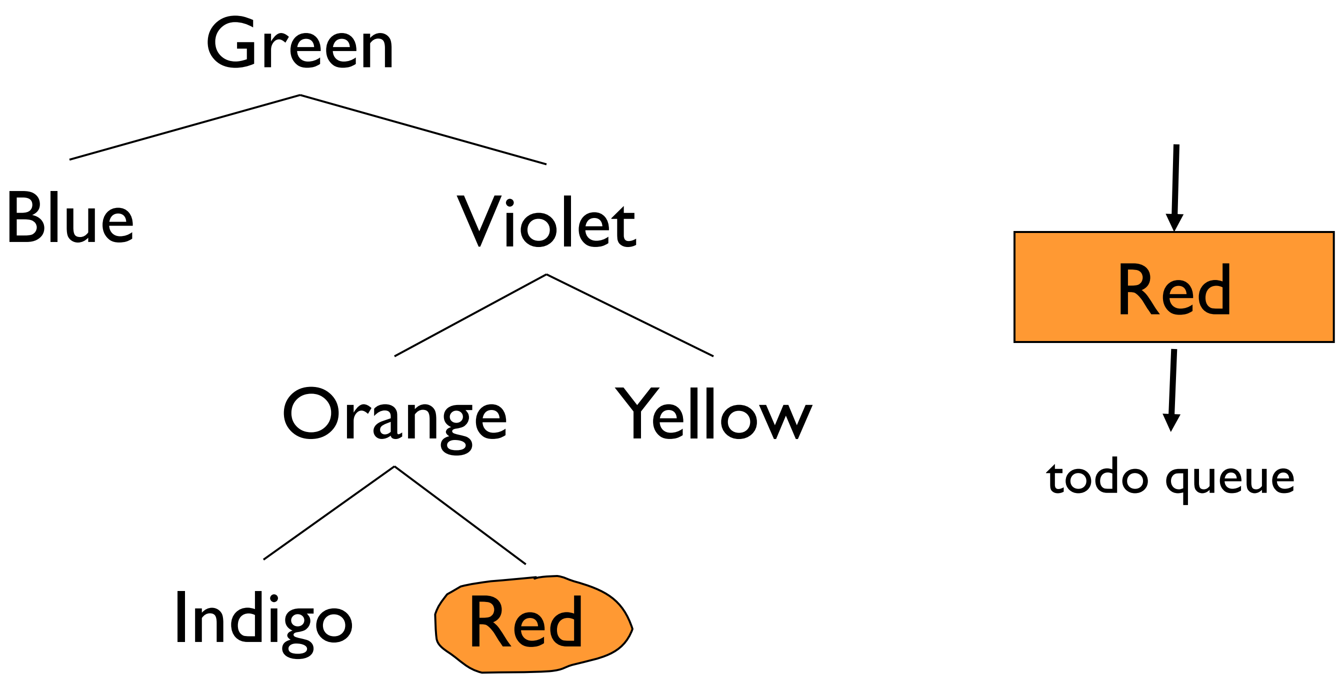
Level-Order Traversal





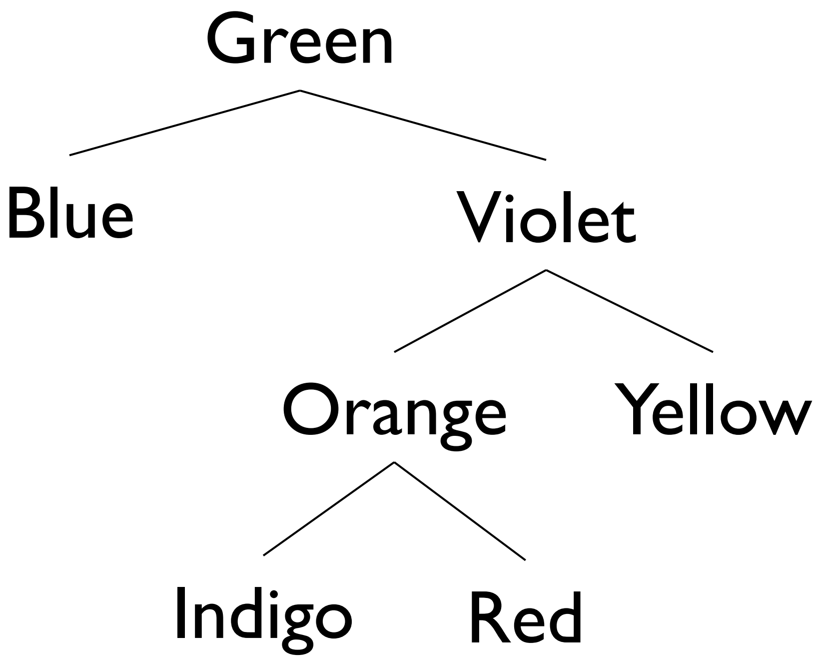
G B V O Y

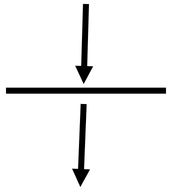
Level-Order Traversal



G B V O Y I

Level-Order Traversal





todo queue

G B V O Y I R

Level-Order Tree Traversal

```
public static <E> void levelOrder(BinaryTree<E> t) {
   if (t.isEmpty()) return;

   // The queue holds nodes for in-order processing
   Queue<BinaryTree<E>> q = new QueueList<BinaryTree<E>>();
   q.enqueue(t); // put root of tree in queue

   while(!q.isEmpty()) {
      BinaryTree<E> next = q.dequeue();
      touch(next);
      if(!next.left().isEmpty()  ) q.enqueue( next.left()  );
      if(!next.right().isEmpty() ) q.enqueue( next.right() );
   }
}
```

Iterators

- Provide iterators that implement the different tree traversal algorithms
- Methods provided by BinaryTree class:
  - preorderIterator()
  - inorderIterator()
  - postorderIterator()
  - levelorderIterator()

Implementing the Iterators

- Basic idea
  - Should return elements in same order as corresponding traversal method shown
  - Recursive methods don’t convert as easily: must phrase in terms of next() and hasNext()
  - So, let’s start with levelOrder!

Level-Order Iterator

```
public BTLevelorderIterator(BinaryTree<E> root)
   {
       todo = new QueueList<BinaryTree<E>>();
       this.root = root; // needed for reset
       reset();
   }

public void reset()
    {
        todo.clear();
        // empty queue, add root
        if (!root.isEmpty()) todo.enqueue(root);
    }
```

Level-Order Iterator

```
public boolean hasNext() {
       return !todo.isEmpty();
}

public E next() {
       BinaryTree<E> current = todo.dequeue();
       E result = current.value();
       if (!current.left().isEmpty())
           todo.enqueue(current.left());
       if (!current.right().isEmpty())
           todo.enqueue(current.right());
       return result;
}
```

Pre-Order Iterator

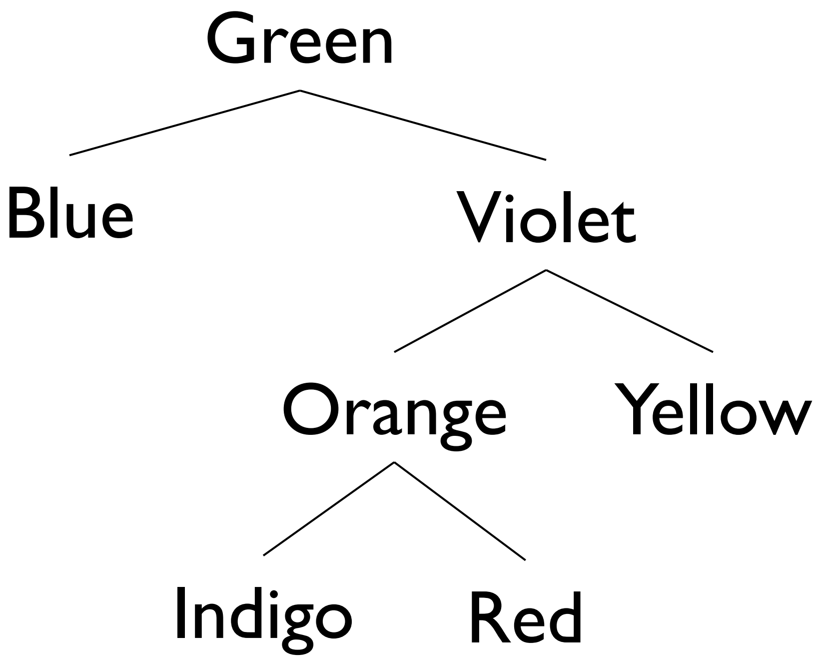
- Basic idea
  - Should return elements in same order as processed by pre-order traversal method
  - Must phrase in terms of next() and hasNext()
  - We “simulate recursion” with stack
    - The stack holds “partially processed” nodes

Pre-Order Iterator

- Outline: node - left tree – right tree
  1. Constructor: Push root onto todo stack
  2. On call to next():
     - Pop node from stack
     - Push right and then left nodes of popped node onto stack
     - Return node's value
  3. On call to hasNext():
     - return !stack.isEmpty()

Pre-Order Iterator

Visit node, then each node in left subtree, then each node in right subtree.



Pre-Order Iterator

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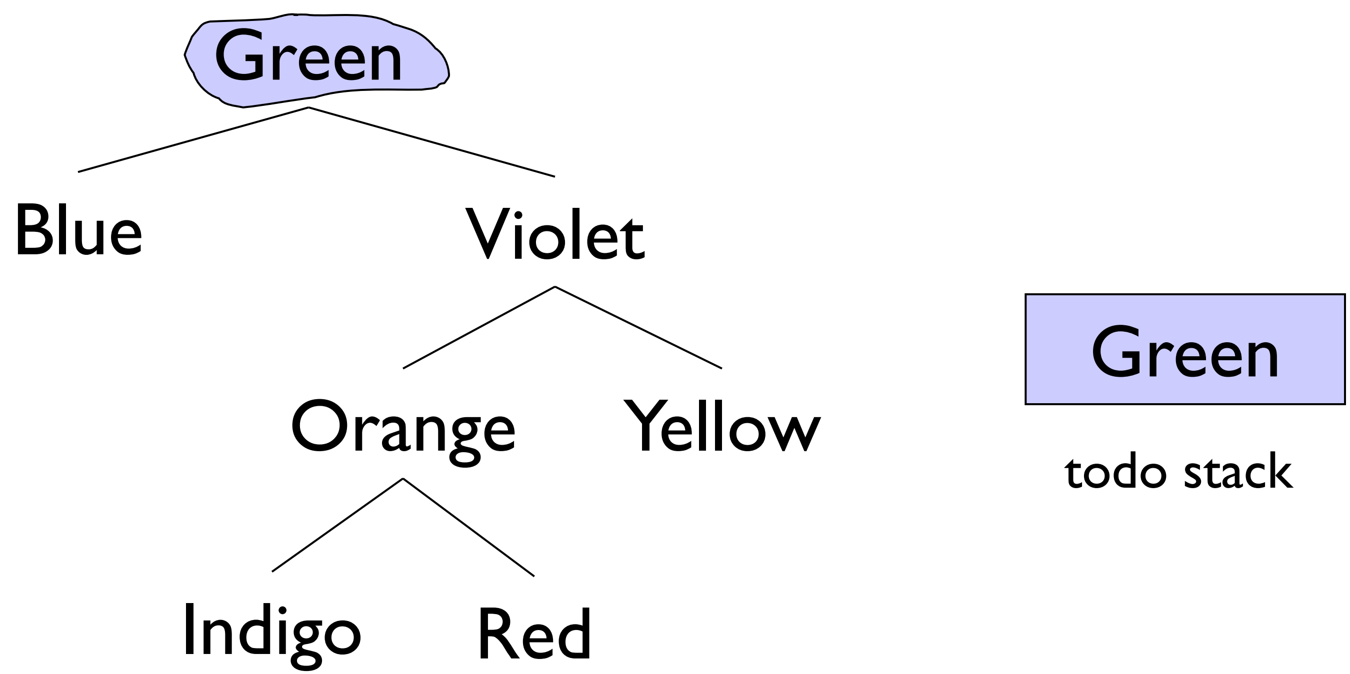

Pre-Order Iterator

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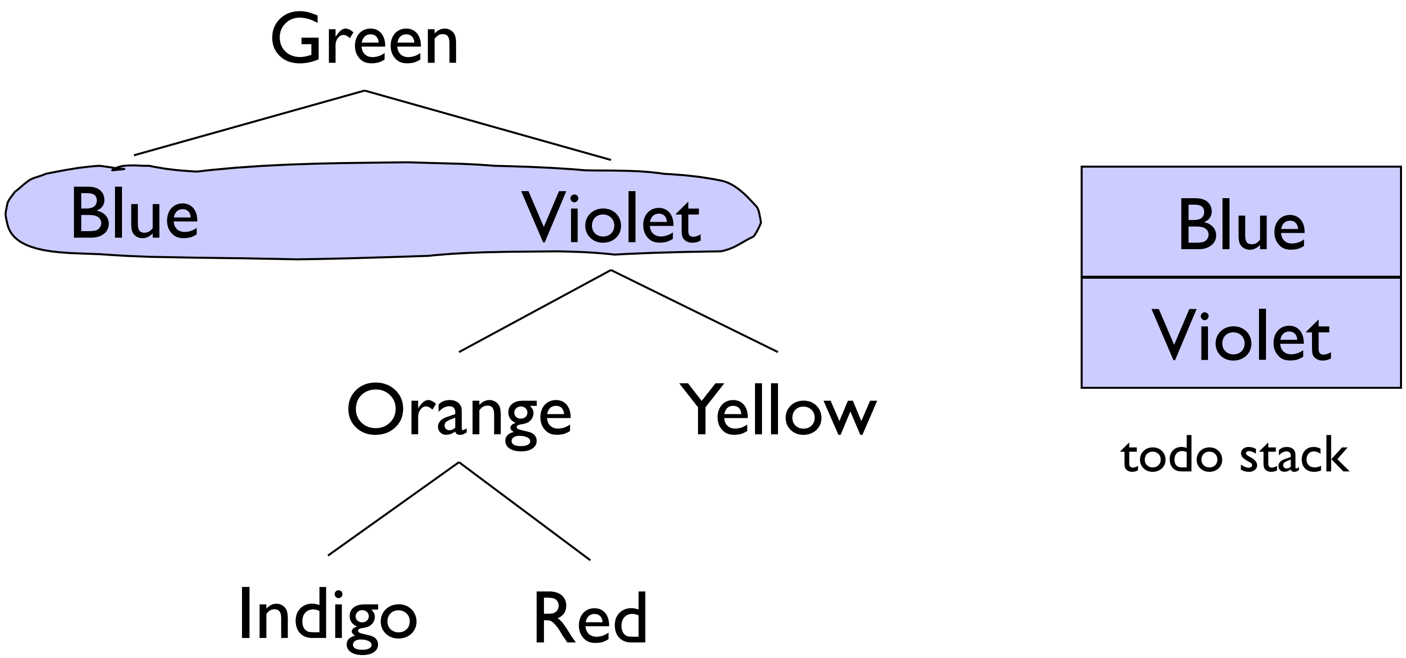

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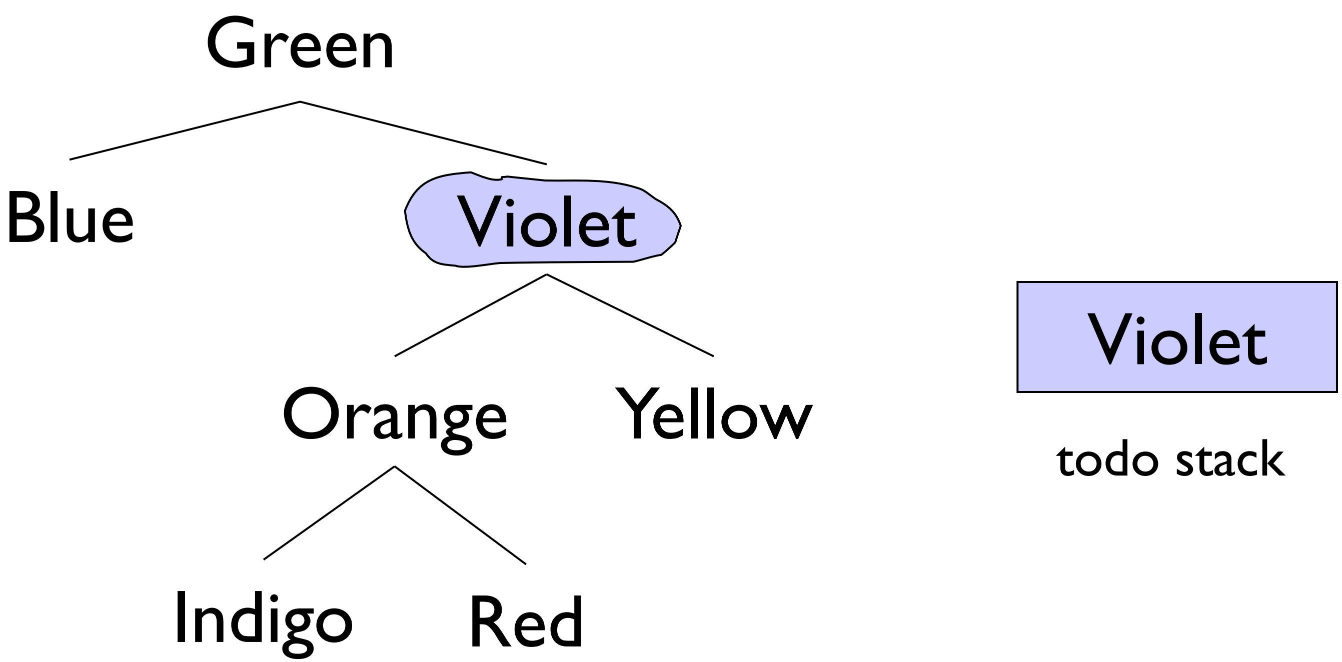
Pre-Order Iterator

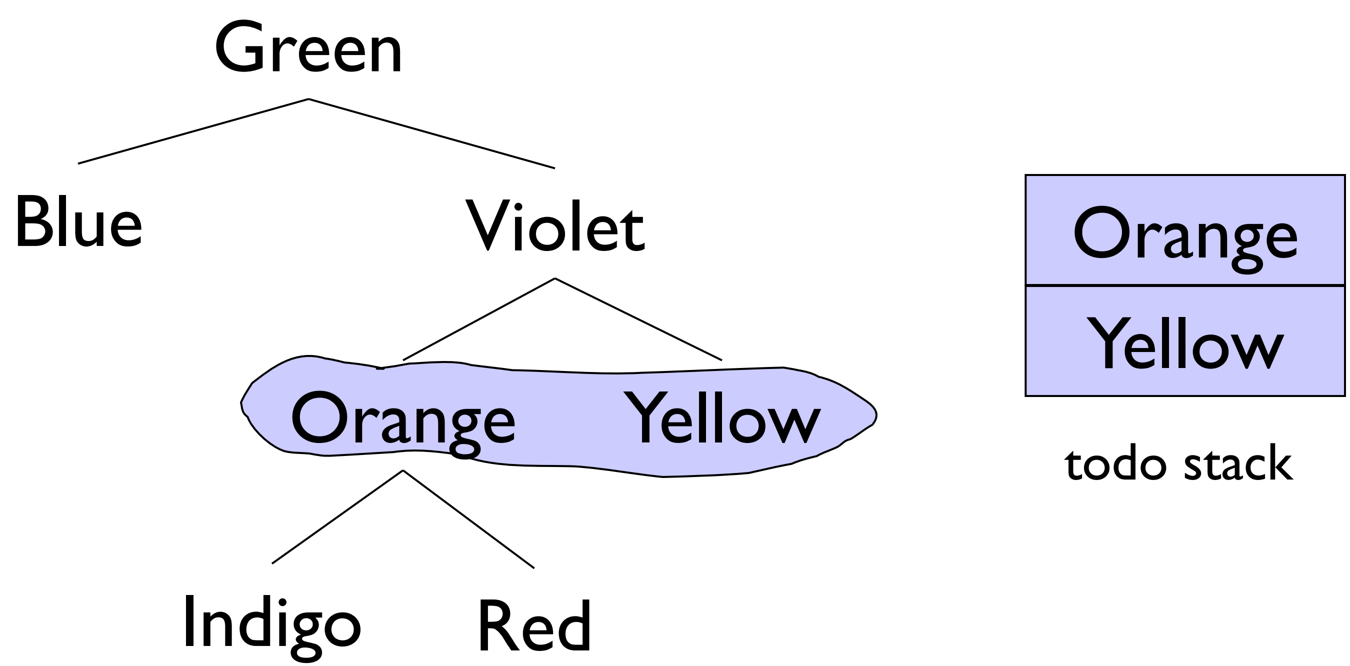
Visit node, then each node in left subtree, then each node in right subtree.



G B

Pre-Order Iterator

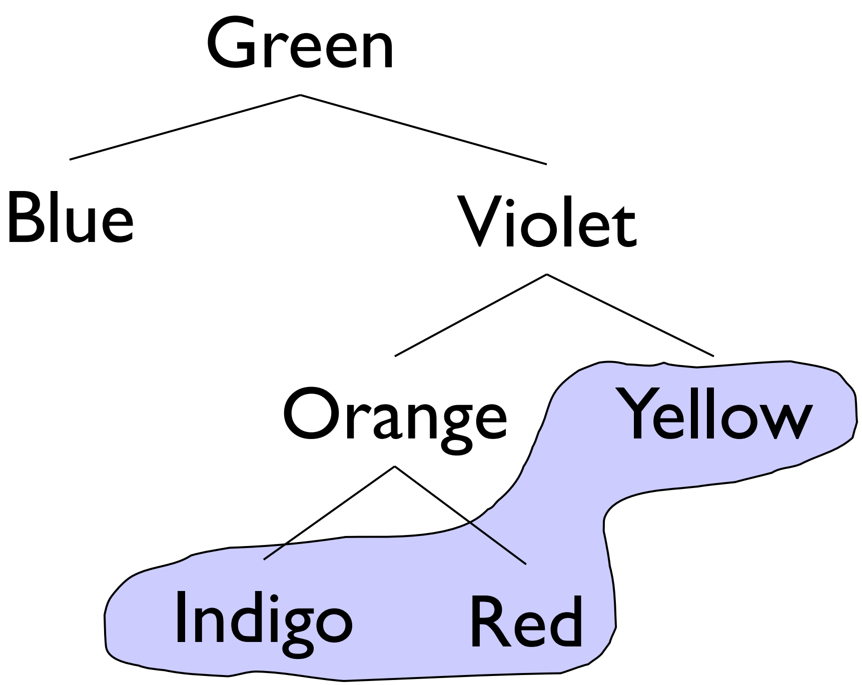
Visit node, then each node in left subtree, then each node in right subtree.

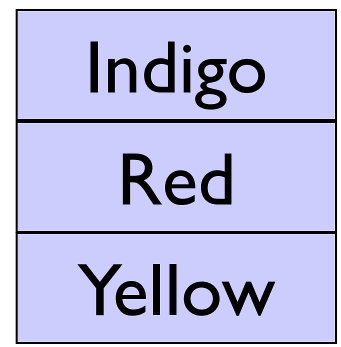


G B V

Pre-Order Iterator

Visit node, then each node in left subtree, then each node in right subtree.



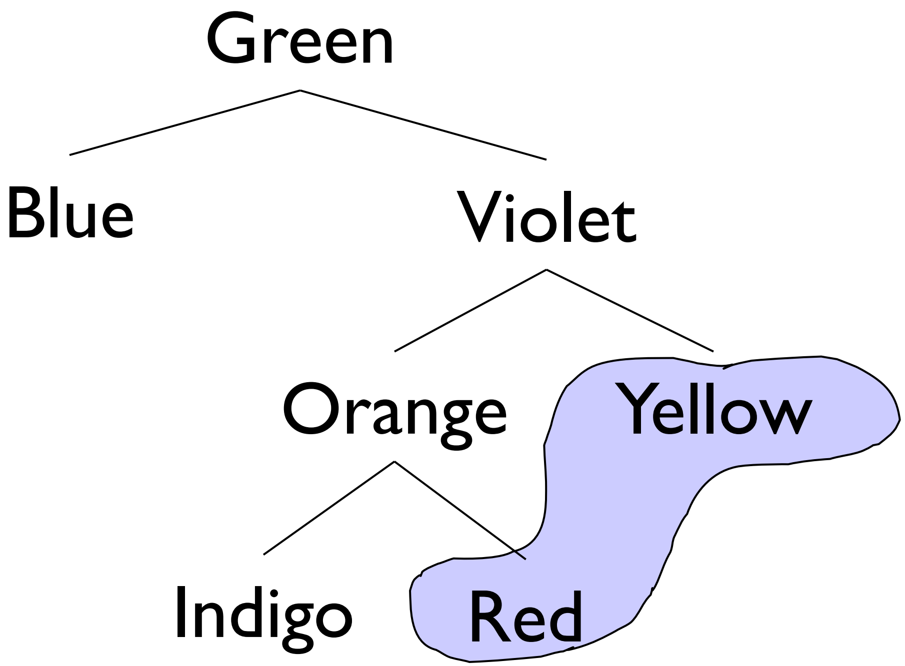


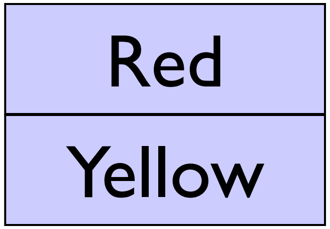
todo stack

G B V O

Pre-Order Iterator

Visit node, then each node in left subtree, then each node in right subtree.



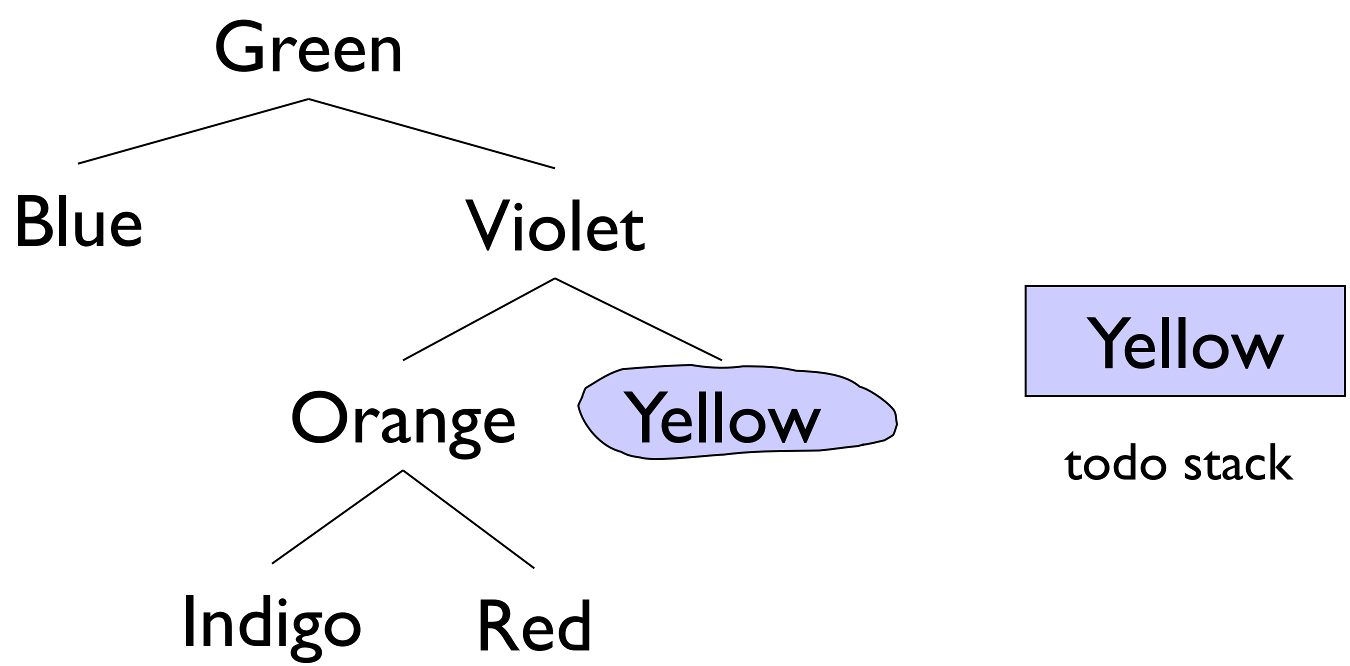


todo stack

G B V O I

Pre-Order Iterator

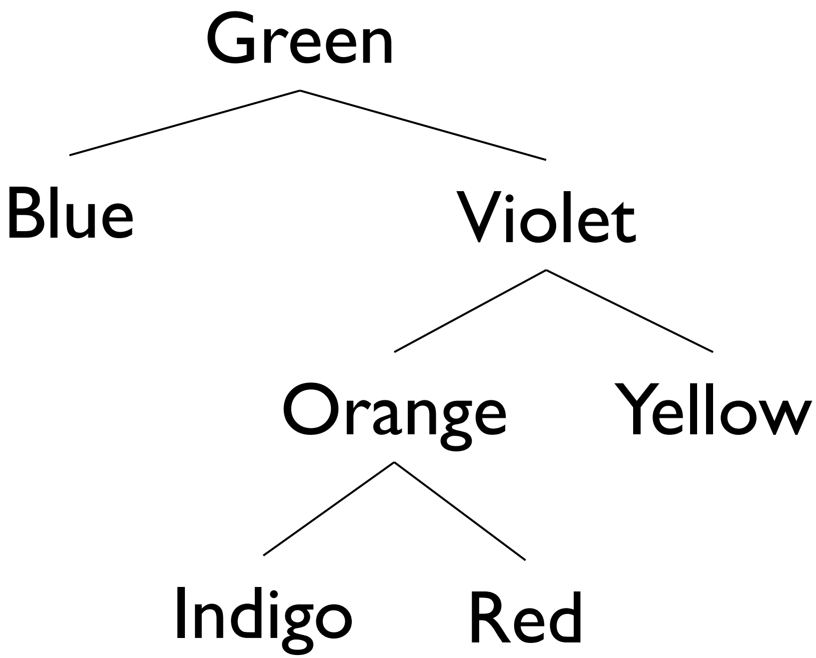
Visit node, then each node in left subtree, then each node in right subtree.



G B V O I R

Pre-Order Iterator

Visit node, then each node in left subtree, then each node in right subtree.



todo stack

G B V O I R Y

Pre-Order Iterator

```
public BTPreorderIterator(BinaryTree<E> root)
   {
       todo = new StackList<BinaryTree<E>>();
       this.root = root;
       reset();
   }

public void reset()
   {
       todo.clear(); // stack is empty; push on root
       if ((!root.isEmpty()) todo.push(root);
   }
```

Pre-Order Iterator

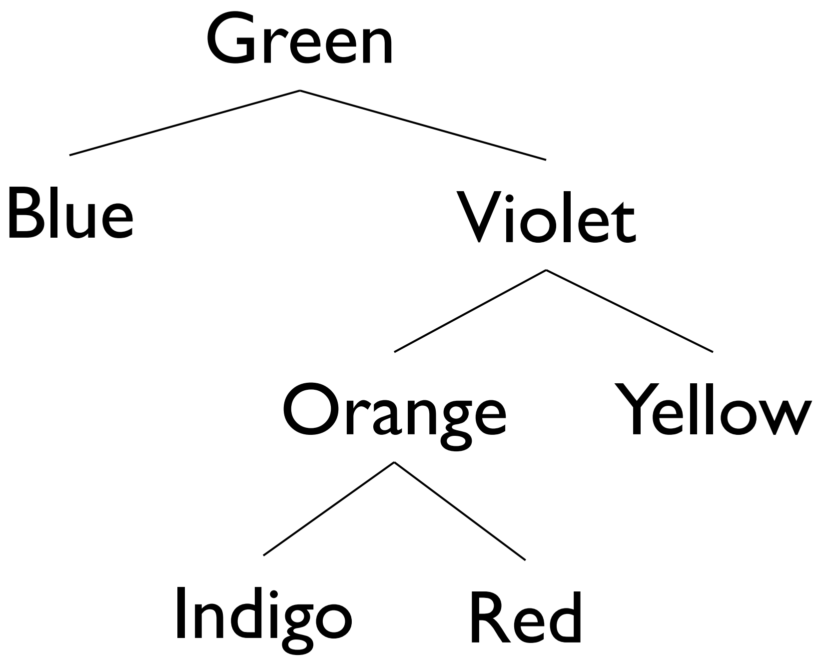
```
public boolean hasNext() {
       return !todo.isEmpty();
}

public E next() {
     BinaryTree<E> old = todo.pop();
     E result = old.value();

     if (!old.right().isEmpty())
           todo.push(old.right());
     if (!old.left().isEmpty())
           todo.push(old.left());
       return result;
   }
```

Tree Traversal Practice Problems

- Prove that levelOrder() is correct: that is, that it touches the nodes of the tree in the correct order (Hint: induction by level)
- Prove that levelOrder() takes O(n) time, where n is the size of the tree
- Prove that the PreOrder (LevelOrder) Iterator visits the nodes in the same order as the PreOrder (LevelOrder) traversal method

In-Order Iterator

Each node is visited after all nodes in left subtree are visited and before any nodes in right subtree.



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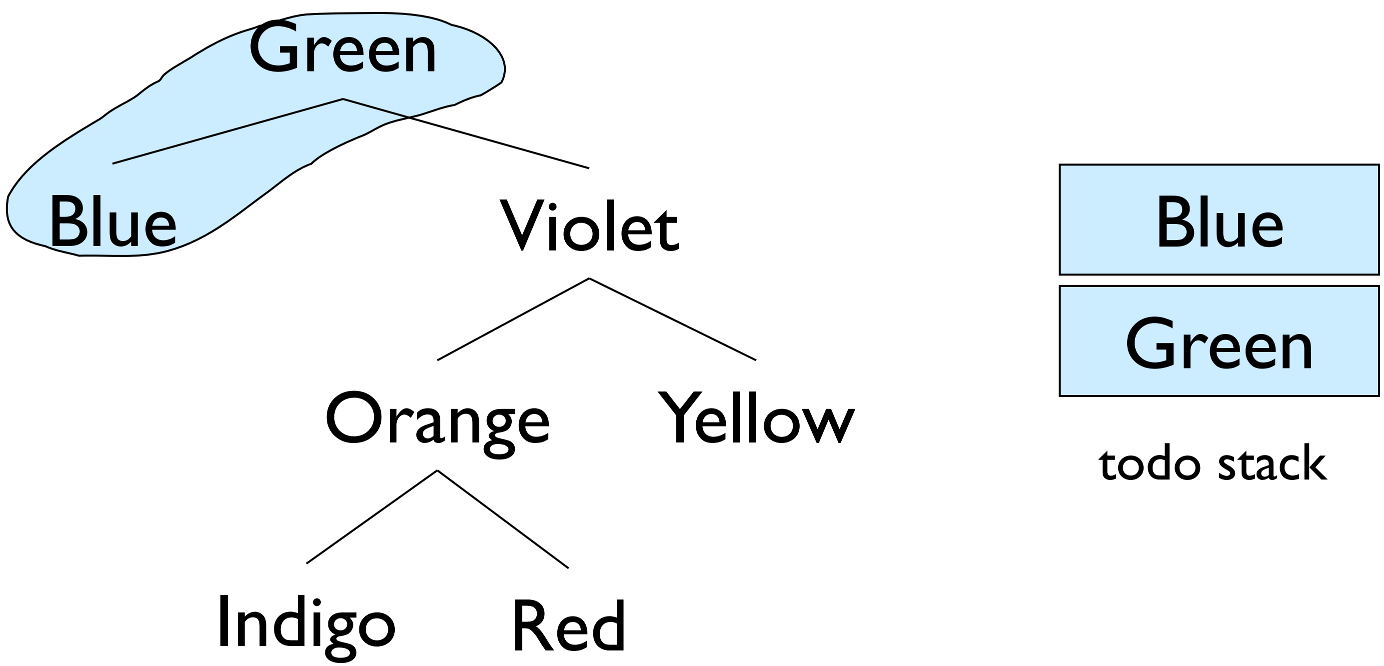

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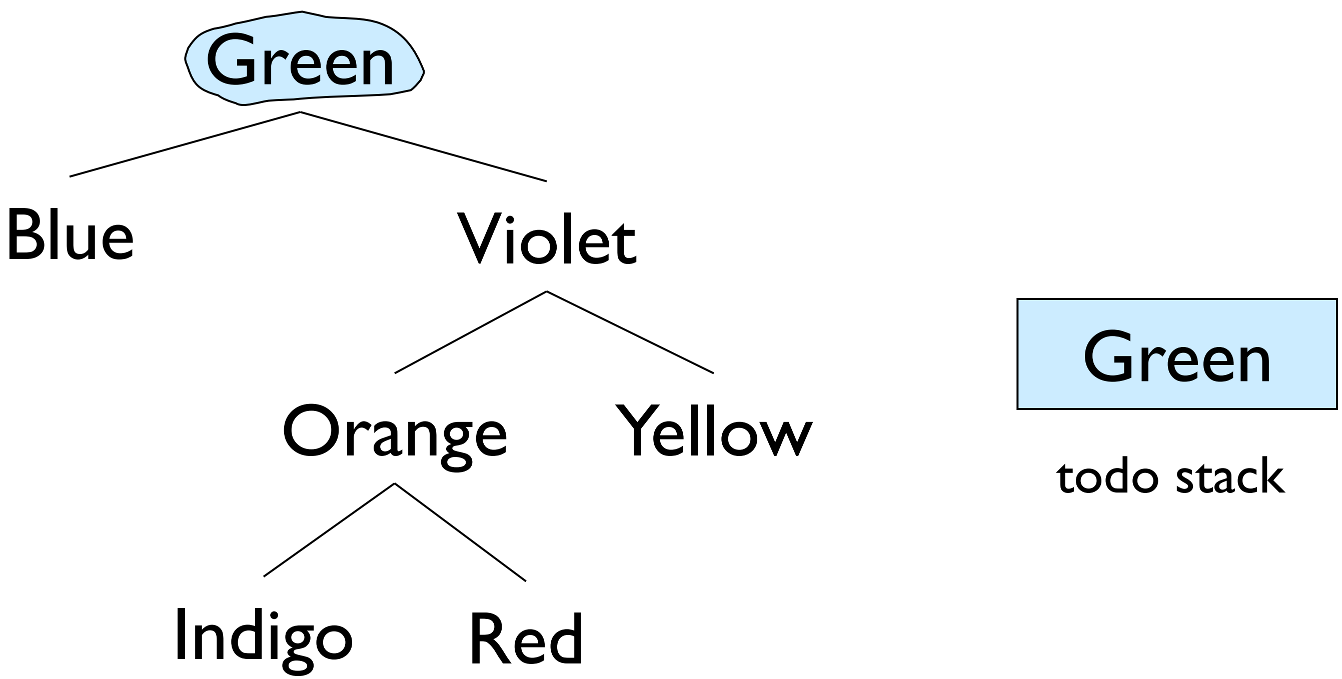

B

In-Order Iterator

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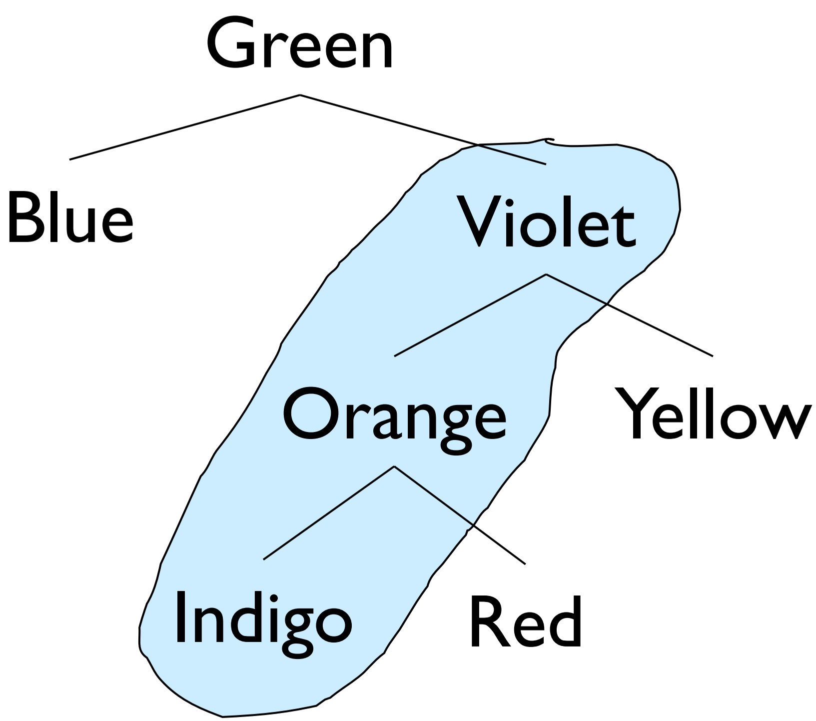

todo stack

B G

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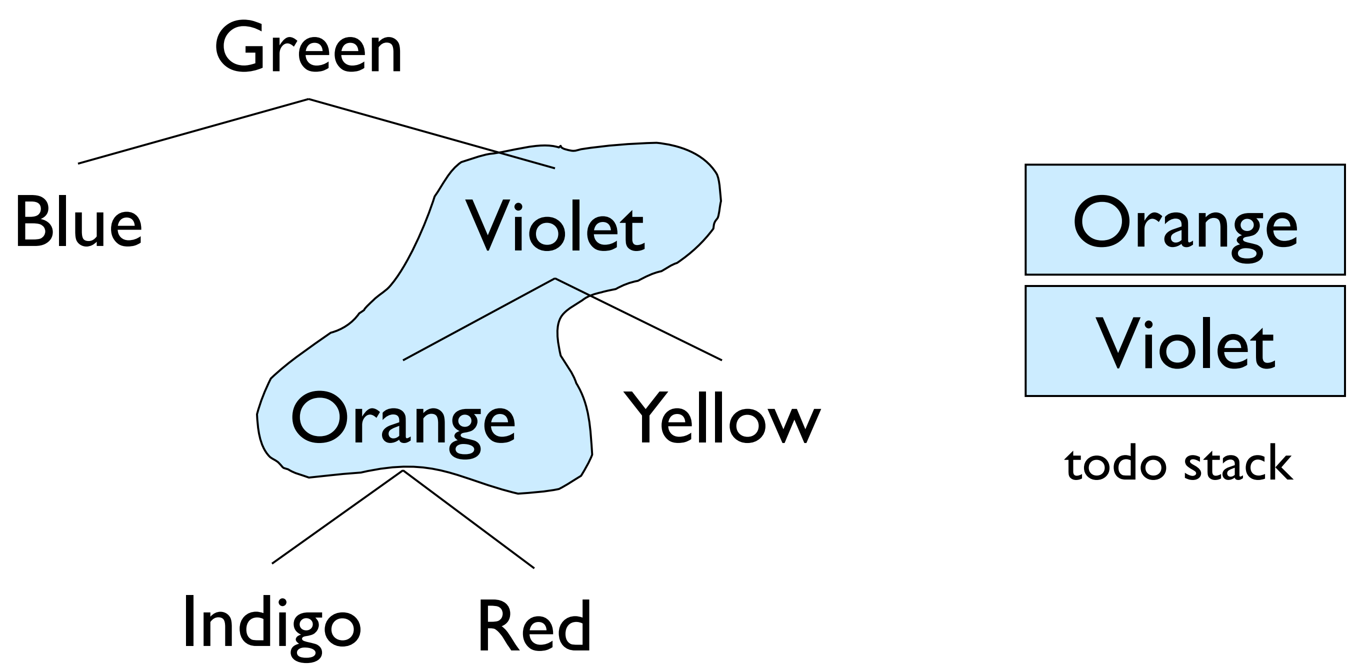

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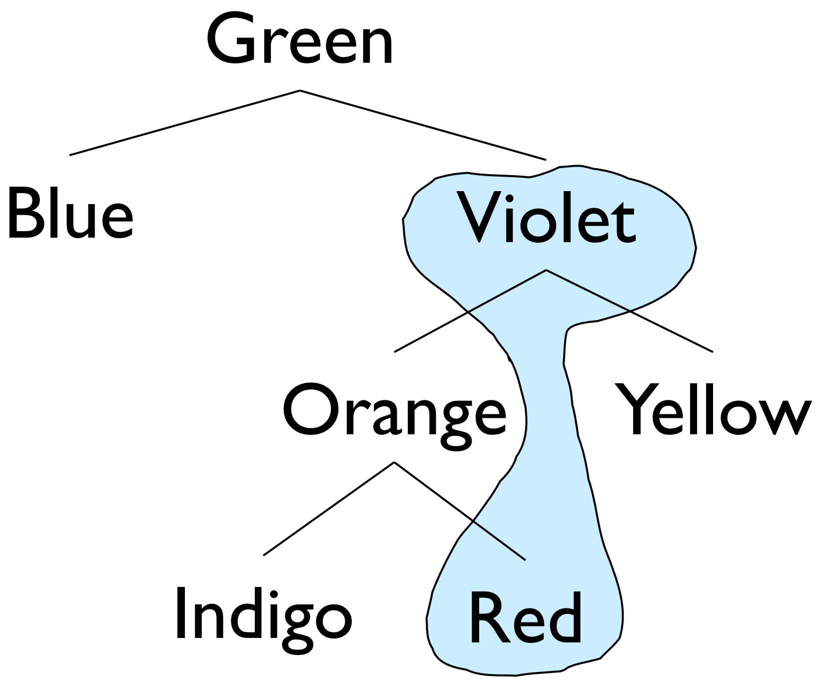

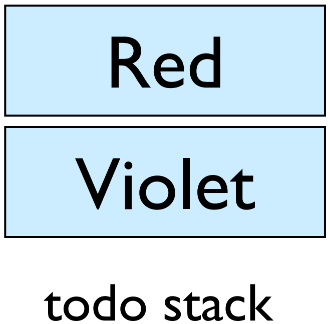

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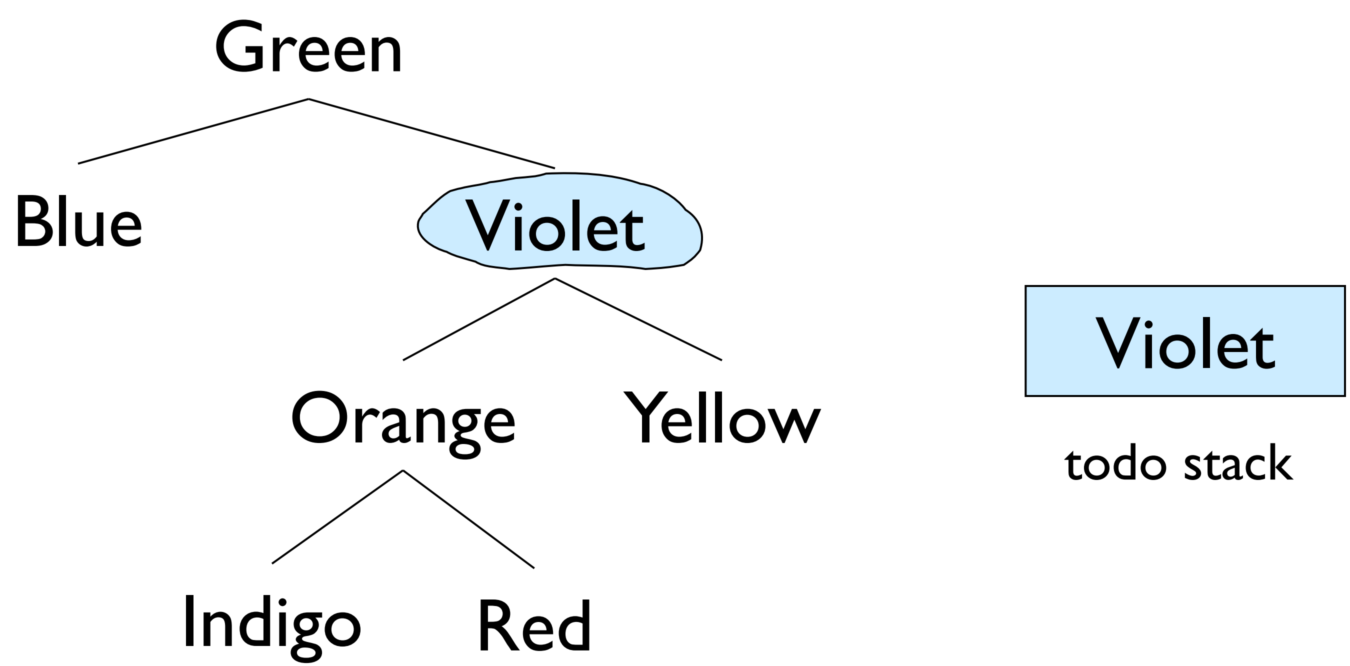

B G I O R

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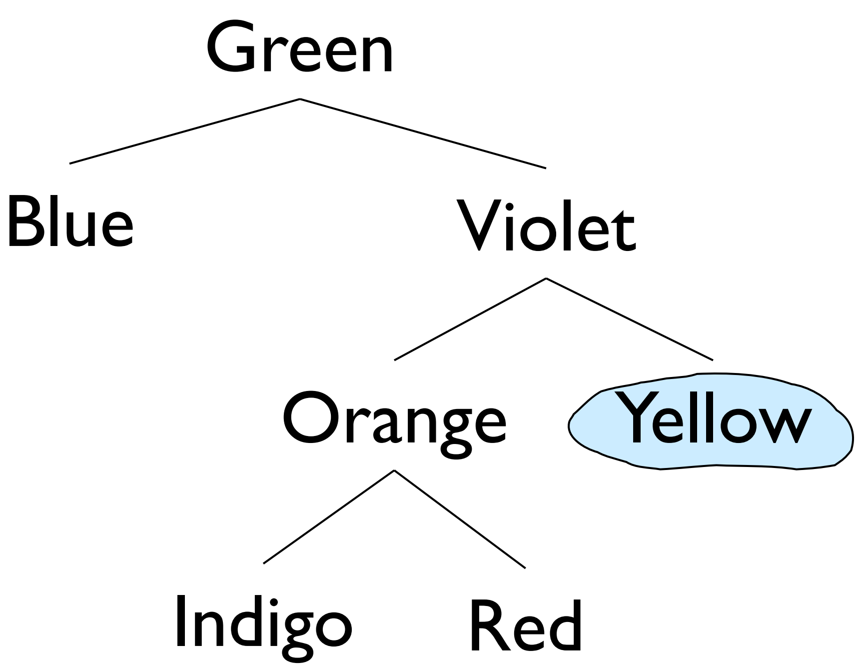

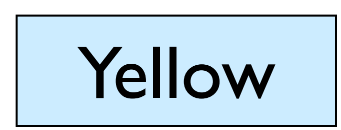

todo stack

B G I O R V

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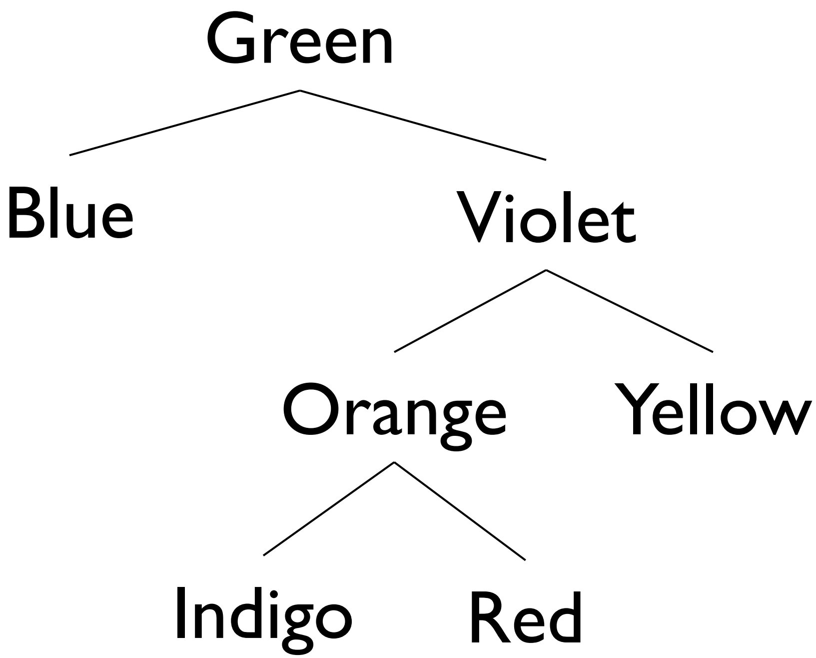


todo stack

B G I O R V Y

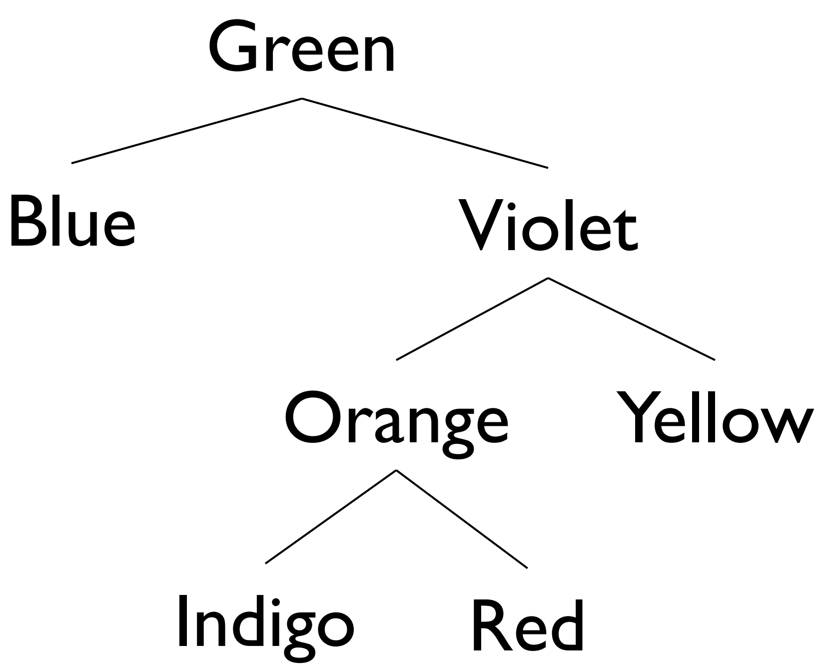
In-Order Iterator

- Outline: left - node - right
  1. Push left children (as far as possible) onto stack
  2. On call to next():
     - Pop node from stack
     - Push right child and follow left children as far as possible
     - Return node's value
  3. On call to hasNext():
     - return !stack.isEmpty()

Post-Order Iterator

- Left as an exercise…

Alternative Tree Representations



- Total # “slots” = 4n
  - Since each BinaryTree maintains a reference to left, right, parent, value
- 2-4x more overhead than vector, SLL, array, …
- But trees capture successor and predecessor relationships that other data structures don’t…

Array-Based Binary Trees

- Encode structure of tree in array indexes
  - Put root at index 0
- Where are children of node i?
  - Children of node i are at 2i+1 and 2i+2
  - Look at example
- Where is parent of node j?
  - Parent of node j is at (j-1)/2

ArrayTree Tradeoffs

- Why are ArrayTrees good?
  - Save space for links
  - No need for additional memory allocated/garbage collected
  - Works well for full or complete trees
    - Complete: All levels except last are full and all gaps are at right
    - "A *complete* binary tree of height h is a full binary tree with 0 or more of the rightmost leaves of level h removed"
- Why bad?
  - Could waste a lot of space
  - Tree of height of n requires 2^{n+1}-1 array slots even if only O(n) elements