CSCI 136
Data Structures &
Advanced Programming

Lecture 18

Fall 2019

Instructor: B&S

Administrative Details

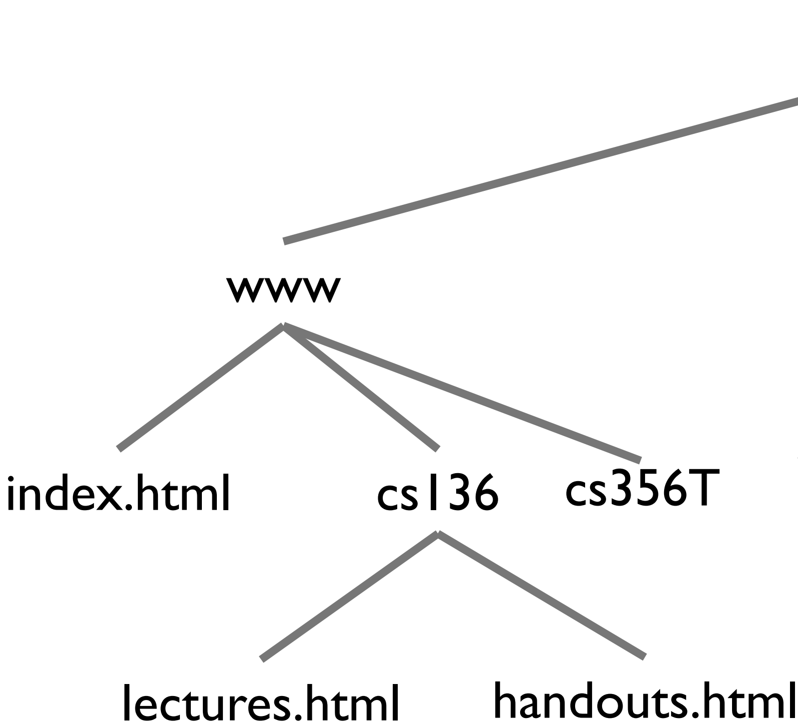
- Lab 6 Today: PostScript
  - No partners this week
  - Review before lab; come to lab with design doc
  - Check out the javadoc pages for the 3 provided classes
    - Token – A wrapper for semantic PS elements,
    - Reader – An iterator to produce a stream of Tokens from standard input or a List of Tokens,
    - SymbolTable – A dictionary with String keys and Token values: For user-defined names

Last Time:

- Ordered Structures

Today

- Trees
  - Structure, Terminology, Examples
  - Implementation
  - Recursion/Induction on Trees
  - Applications
  - Traversals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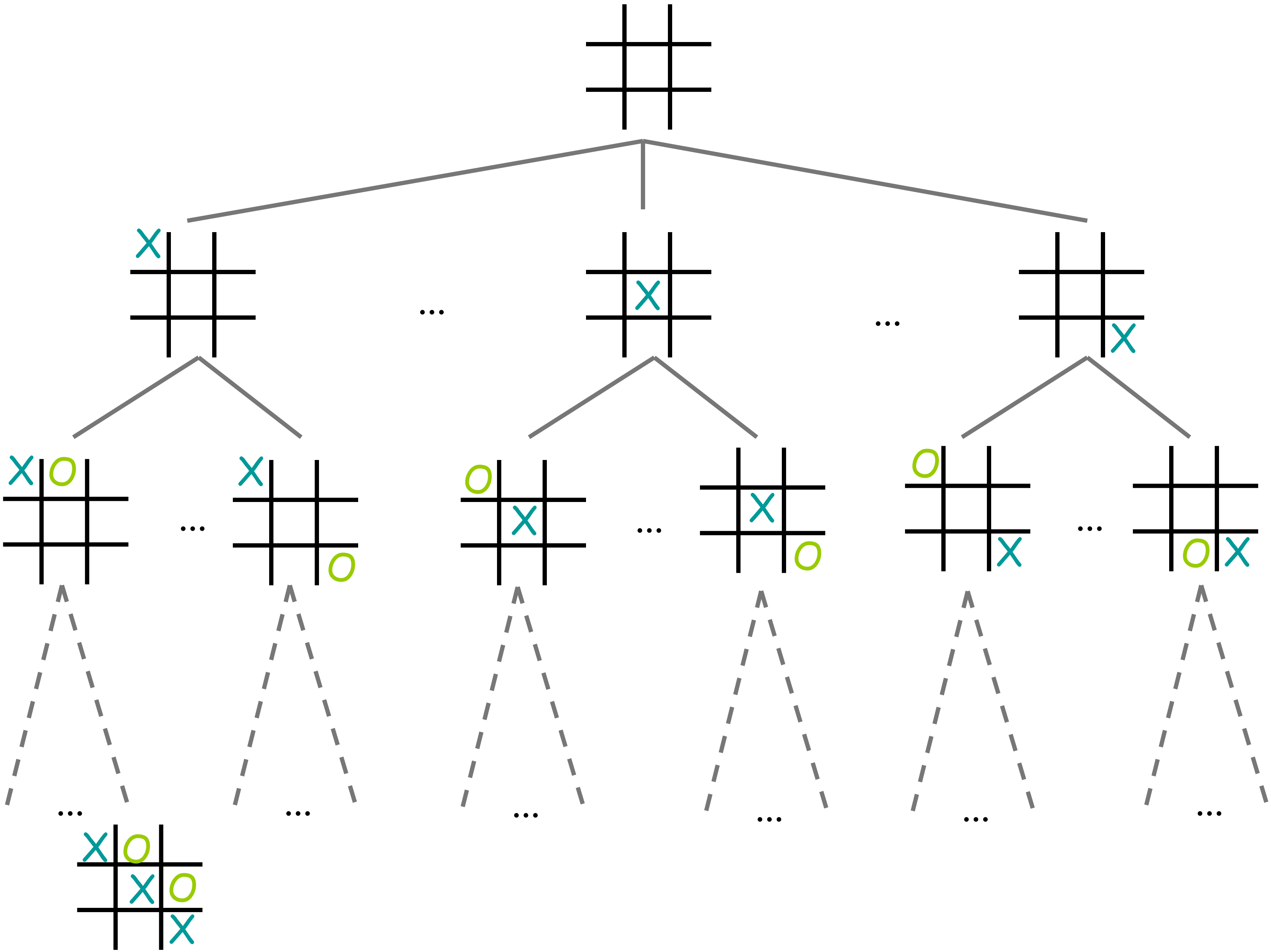
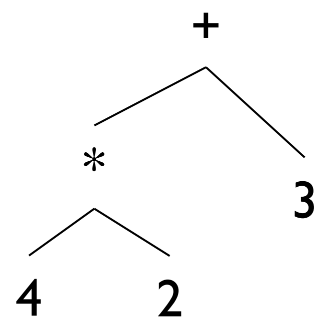


Tree Features

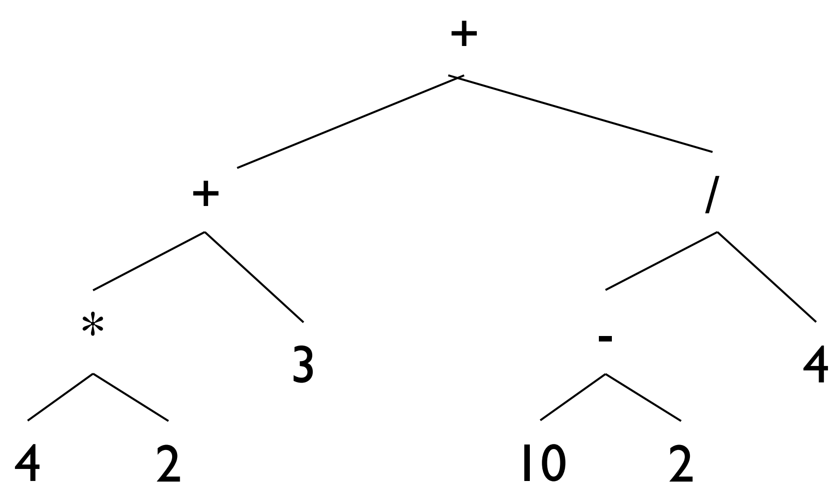
- Hierarchical relationship
- Root at the top
- Leaf at the bottom
- Interior nodes in middle
- Parents, children, ancestors, descendants, siblings
- Degree (of node): number of children of node
- Degree (of tree): maximum degree (across all nodes)
- Depth of node: number of *edges* from root to node
- Height of tree: maximum depth (across all nodes)

Expression Trees

4 * 2 + 3



(4 * 2 + 3) + ((10 – 2)/ 4)

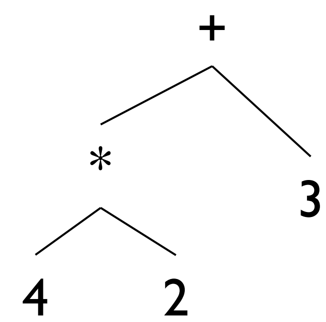


Introducing Binary Trees

- Degree of each node at most 2
- Recursive nature of tree
  - Empty
  - Root with left and right subtrees
- SLL: Recursive nature was captured by hidden node (Node<E>) class
- Binary Tree: No “inner” node class
  - Single BinaryTree class does it all
  - Is it a tree or a node?
    - It’s a node that’s a root of a tree!
  - And it’s not part of Structure hierarchy!

Expression Trees

4 * 2 + 3



Build using constructor

```
new BinaryTree<E>(value, leftSubTree, rightSubTree)
```

```
BinaryTree<String> fourTimesTwo =  new BinaryTree<String>
        (“*”,new BinaryTree<String>(“4”),new BinaryTree<String>(“2”));

BinaryTree<String> fourTimesTwoPlusThree = new BinaryTree<String>
        (“+”, fourTimesTwo,  new BinaryTree<String>(“3”));
```

Expression Trees

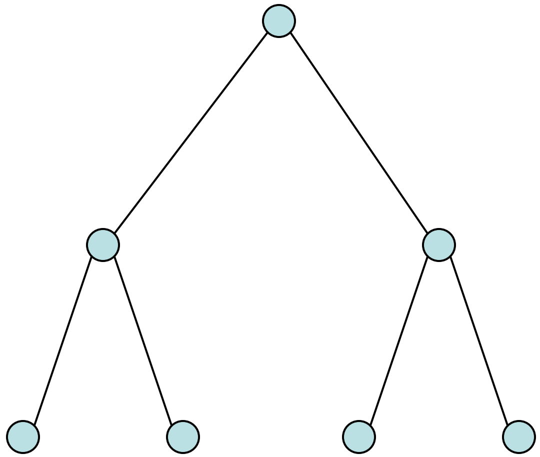
- General strategy
  - Make a binary tree (BT) for each leaf node
  - Move from bottom to top, creating BTs
  - Eventually reach the root
  - Call "evaluate" on final BT

- Example
  - How do we make a binary expression tree for $(((4+3)*(10-5))/2)$
    - Postfix notation: 4 3 + 10 5 - * 2 /

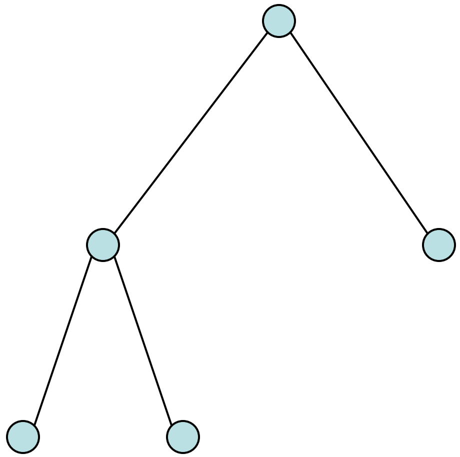
```
int evaluate(BinaryTree<String> expr) {

    if (expr.height() == 0)
      return Integer.parseInt(expr.value());

    else {
      int left = evaluate(expr.left());
      int right = evaluate(expr.right());
      String op = expr.value();
      switch (op) {
      case "+" : return left + right;
      case "-" : return left - right;
      case "*" : return left * right;
      case "/" : return left / right;
      }
      Assert.fail("Bad op");
      return -1;
   }
}
```

Full vs. Complete (non-standard!)

- Full tree – A full binary tree of height h has *leaves only* on level h, and each internal node has exactly 2 children.



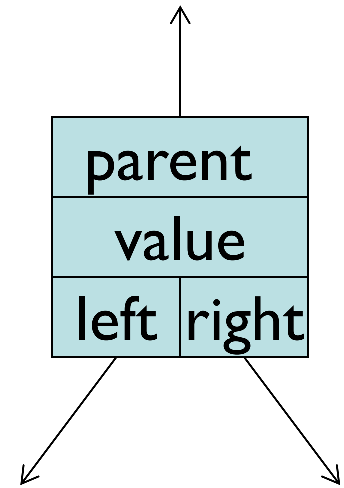
- Complete tree – A *complete* binary tree of height h is *full* to height h-1 and has all leaves at level h in leftmost locations.

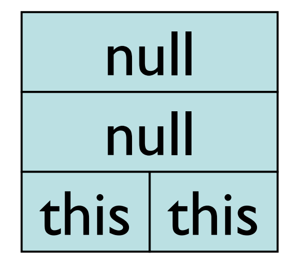


All full trees are complete, but not all complete trees are full!

Implementing BinaryTree

- BinaryTree<E> class
  - Instance variables
    - BinaryTree: parent, left, right
    - E: value
- left and right are never null
  - If no child, they point to an "empty" tree
    - Empty tree T has value null, parent null, left = right = T
  - Only empty tree nodes have null value

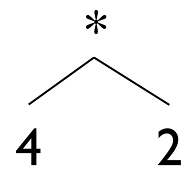


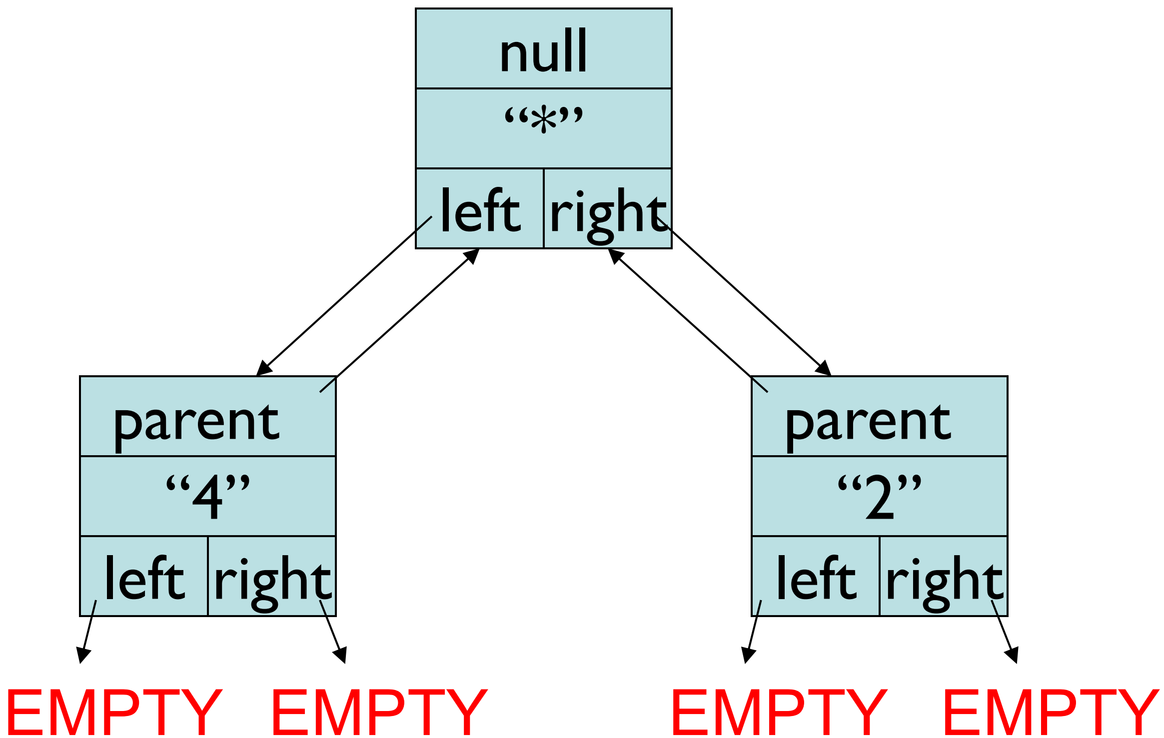


EMPTY BT

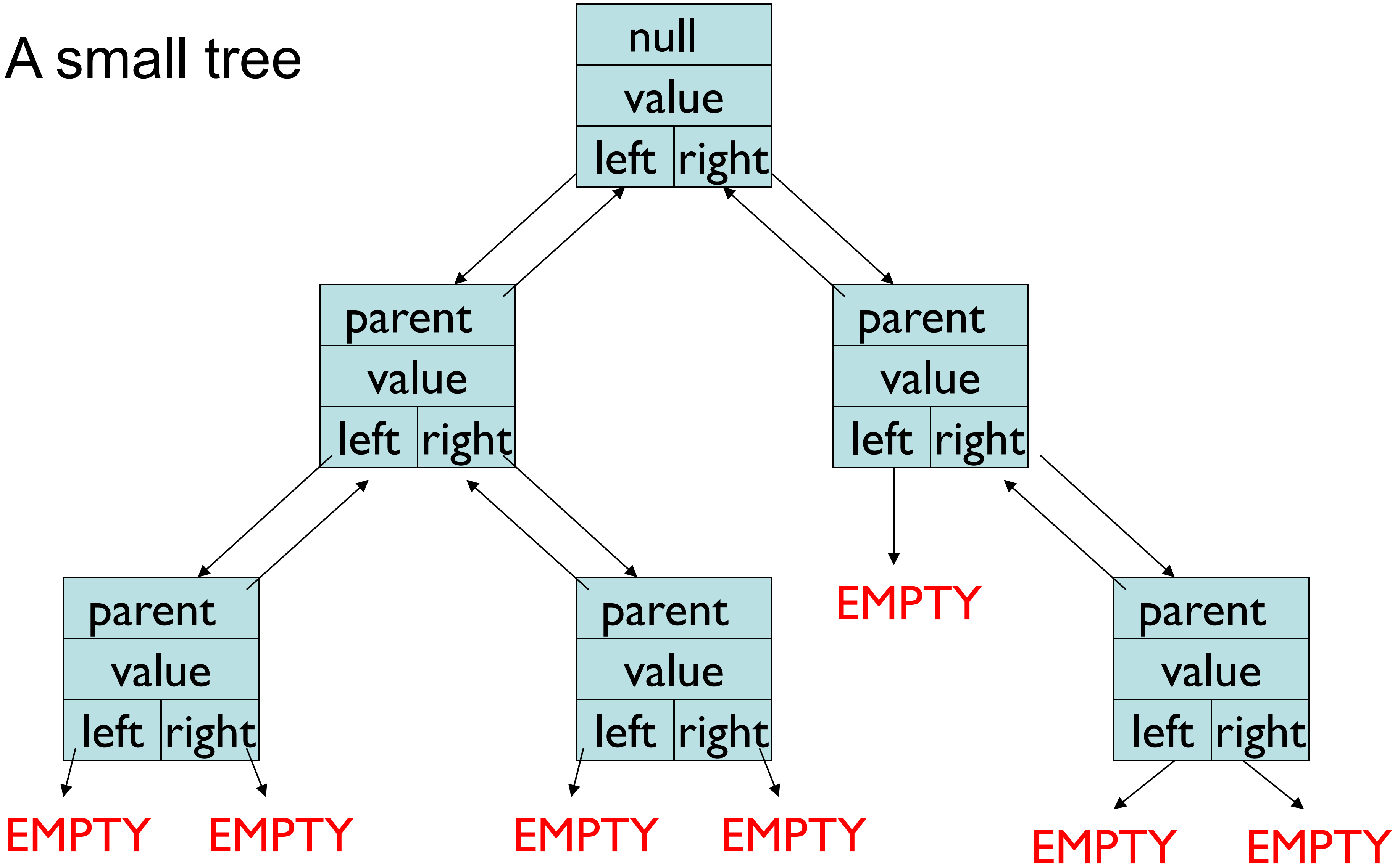
Implementing BinaryTree

- BinaryTree class
  - Instance variables
    - BT parent, BT left, BT right, E value





A small tree



EMPTY != null!

Implementing BinaryTree

- Many (!) methods: See BinaryTree javadoc page
- All "left" methods have equivalent "right" methods
  - public BinaryTree()
    - // generates an empty node (EMPTY)
    - // parent and value are null, left=right=this
  - public BinaryTree(E value)
    - // generates a tree with a non-null value and two empty (EMPTY) subtrees
  - public BinaryTree(E value, BinaryTree<E> left, BinaryTree<E> right)
    - // returns a tree with a non-null value and two subtrees
  - public void setLeft(BinaryTree<E> newLeft)
    - // sets left subtree to newLeft
    - // re-parents newLeft by calling newLeft.setParent(this)
  - protected void setParent(BinaryTree<E> newParent)
    - // sets parent subtree to newParent
    - // called from setLeft and setRight to keep all "links" consistent

Implementing BinaryTree

- Methods:
  - public BinaryTree<E> left()
    - // returns left subtree
  - public BinaryTree<E> parent()
    - // post: returns reference to parent node, or null
  - public boolean isLeftChild()
    - // returns true if this is a left child of parent
  - public E value()
    - // returns value associated with this node
  - public void setValue(E value)
    - // sets the value associated with this node
  - public int size()
    - // returns number of (non-empty) nodes in tree
  - public int height()
    - // returns height of tree rooted at this node
  - But where's "remove" or "add"?!?!

BT Questions/Proofs

- Prove
  - The number of nodes at depth n is at most 2^n.
  - The number of nodes in tree of height n is at most $2^{(n+1)}-1$.
  - A tree with n nodes has exactly n-1 edges
  - The size() method works correctly
  - The height() method works correctly
  - The isFull() method works correctly

BT Questions/Proofs

Prove: Number of nodes at depth $d \geq 0$ is at most 2^d.

Idea: Induction on depth d of nodes of tree

Base case: $d = 0$: 1 node. $1 = 2^0$ ✓

Induction Hyp.: For some $d \geq 0$, there are at most 2^d nodes at depth d.

Induction Step: Consider depth $d+1$. There are at most 2 nodes at depth $d+1$ for every node at depth d.

Therefore it has at most $2 * 2^d = 2^{d+1}$ nodes ✓

BT Questions/Proofs

Prove that any tree on $n \geq 1$ nodes has $n-1$ edges

Idea: Induction on number of nodes

Base case: $n = 1$. There are no edges ✓

Induction Hyp: Assume that, for some $n \geq 1$, every tree on n nodes has exactly $n-1$ edges.

Induction Step: Let T have $n+1$ nodes. Show it has exactly n edges.

- Remove a leaf v (and its single edge) from T
- Now T has n nodes, so it has $n-1$ edges
- Now add v (and its single edge) back, giving $n+1$ nodes and n edges.

BT Questions/Proofs

Prove that BinaryTree method size() is correct.

- Let n be the number of nodes in the tree T
- Alert: Strong Induction Ahead...

Base case: n = 0. T is empty---size() returns 0✓

Induction Hyp: Assume size() is correct for *all trees* having *at most* n nodes.

Induction Step: Assume T has n+1 nodes

- Then left/right subtrees each have *at most* n nodes
- So size() returns correct value for each subtree
- And the size of T is 1 + size of left subtree + size of right subtree✓

Representing Knowledge

- Trees can be used to represent knowledge
  - Example: InfiniteQuestions game
- We often call these trees decision trees
  - Leaf: object
  - Internal node: question to distinguish objects
- Move down decision tree until we reach a leaf node
- Check to see if the leaf is correct
  - If not, add another question, make new and old objects children
- Let's look at the code…

Building Decision Trees

- Gather/obtain data
- Analyze data
  - Make greedy choices: Find good questions that divide data into halves (or as close as possible)
- Construct tree with shortest height
- In general this is a *hard* problem!
- Example

Representing Arbitrary Trees

- What if nodes can have many children?
  - Example: Game trees
- Replace left/right node references with a list of children (Vector, SLL, etc)
  - Allows getting "i^{th}" child
- Should provide method for getting degree of a node
- Degree 0 Empty list No children Leaf